Supplementary appendices for 'Agenda-manipulation in ranking'

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E Proofs of Propositions 1 and 2 (§6)

In this appendix, we establish tightness for the characterisations of regret-freeness in Theorems 2 and 3. We begin in §E.1 with a lemma, then use it to deduce Proposition 2 (§E.2) and Proposition 1 (§E.3).

E.1 A lemma

Definition 10. Given a proto-ranking R and alternatives $x \succ y$ and $z \neq w$, say that $\{z, w\}$ *makes* $\{x, y\}$ *an error at* R iff both $x \not R\, y \not R\, x$ and $z \not R\, w \not R\, z$, and one of the following holds:

- $x \succ z \succ y$, $y \not R\, z \not R\, x$, and $w \in \{x, y\}$.
- $z \succ y$, $x \mathrel{R} z$ and $w = y$.
- $x \succ z$, $z \mathrel{R} y$ and $w = x$.

If $\{z, w\}$ makes $\{x, y\}$ an error, then offering $\{x, y\}$ either misses an opportunity or takes a risk at R, and the chair 'should' offer $\{z, w\}$ instead.[39]

Recall from appendix C.1 the definition of a missed opportunity.

Lemma 8. Let R be a proto-ranking, and let $A \subseteq \mathcal{X}^2$ be a non-empty set of pairs of distinct alternatives. Suppose that for any pair $\{x, y\} \in A$, there is a pair $\{z, w\} \in A$ that makes $\{x, y\}$ an error at R. Then R contains a missed opportunity.

Proof. Let R and A satisfy the hypothesis. Then there is a pair $\{z, w\} \in A$ and another pair $\{z', w'\} \in A$ that makes $\{z, w\}$ an error at R. Assume (wlog) that $z \succ w$ and $z' \succ w'$. Since $\{z, w\} \neq \{z', w'\}$, we must have either $z \neq z'$ or $w \neq w'$. Assume that $z \neq z'$; the case $w \neq w'$ is similar.

First claim. There exists a sequence $(x_t)_{t=1}^T$ in $\mathcal{X}$ with $T \geq 2$ and $x_1 \neq x_2$ such that for every $t \leq T$, writing $x_{T+1} \coloneqq x_1$,

[39] This is heuristic, as offering $\{z, w\}$ might itself miss an opportunity or take a risk at R.

(i) if $x_t \succ x_{t+1}$ then $x_{t+1} \not R\, x_t$, and

(ii) if $x_{t+1} \succ x_t$ then $x_t \, R \, x_{t+1}$.

Proof of the first claim. Define $\{x_1, y_1\} \coloneqq \{z, w\}$ and $\{x_2, y_2\} \coloneqq \{z', w'\}$. By the hypothesis of the lemma, there is a pair $\{x_3, y_3\} \in A$ with (wlog) $x_3 \succ y_3$ that makes $\{x_2, y_2\}$ an error at R, a $\{x_4, y_4\} \in A$ with $x_4 \succ y_4$ that makes $\{x_3, y_3\}$ an error at R, and so on. Since A is finite, $\{x_1, y_1\}$ makes $\{x_T, y_T\}$ an error for some $T \in \mathbf{N}$. We have $T \geq 2$ and $x_1 \neq x_2$ by construction, and (i)–(ii) must hold because $\{x_{t+1}, y_{t+1}\}$ makes $\{x_t, y_t\}$ an error at R. □

Let $(x_t)_{t=1}^T$ be a minimal sequence satisfying the conditions of the first claim (one with no strict subsequence that satisfies the conditions).

Second claim. $x_t \neq x_s$ for all distinct $t, s \in \{1, \ldots, T\}$.

Proof of the second claim. Suppose toward a contradiction that $x_t = x_{t+1}$; then the sequence $x_1, \ldots, x_{t-1}, x_{t+1}, \ldots, x_T$ satisfies the conditions of the first claim, contradicting the minimality of $(x_t)_{t=1}^T$. Assume for the remainder that $x_t \neq x_{t+1}$ for every $t \in \{1, \ldots, T\}$.

Suppose toward a contradiction that $x_t = x_s$, where $t + 1 < s$. Then the sequence $x_{t+1}, \ldots, x_s$ satisfies the conditions of the first claim, which is absurd since $(x_t)_{t=1}^T$ is minimal. □

In light of the second claim, we may re-label the sequence $(x_t)_{t=1}^T$ so that $x_1 \succ x_t$ for every $t \in \{2, \ldots, T\}$. Let $t' \leq T$ be the least $t \geq 2$ such that $x_T \succ x_{T-1} \succ \cdots \succ x_t$. (So $t' = T$ exactly if $x_{T-1} \succ x_T$.) We shall show that $\{x_1, x_{t'}\}$ is a missed opportunity in R; in particular, that $t' \geq 3$ and

(a) $x_1 \succ x_{t'-1} \succ x_{t'}$,

(b) $x_{t'} \, R \, x_1$, and

(c) $x_{t'} \not R \, x_{t'-1} \not R \, x_1$.

For (b), if $t' = T$ then $x_{t'} = x_T \, R \, x_1$ by property (ii), and if not then $x_{t'} \, R \cdots R \, x_T \, R \, x_1$ by property (ii), whence $x_{t'} \, R \, x_1$ by transitivity of R. The second half of (a) (i.e. $x_{t'-1} \succ x_{t'}$) holds by definition of t'. The first half of (c) (i.e. $x_{t'} \not R \, x_{t'-1}$) then follows by property (i). Since $x_{t'} \, R \, x_1$, it follows that $t' - 1 \neq 1$, which is to say that $t' \geq 3$. The first half of (a) (i.e. $x_1 \succ x_{t'-1}$) then holds by construction. Finally, the second half of (c) (i.e. $x_{t'-1} \not R \, x_1$) must hold since otherwise the sequence $(x_t)_{t=1}^{t'-1}$ would satisfy the conditions of the first claim, contradicting the minimality of $(x_t)_{t=1}^T$. ■

E.2 Proof of Proposition 2 (p. 14)

At a history at which the chair has committed no errors, the proto-ranking clearly contains no missed opportunities. The following therefore implies Proposition 2.

Proposition 2⋆. Let R be a non-total proto-ranking containing no missed opportunities. Then there exist distinct $x, y \in \mathcal{X}$ such that $x \not R\, y \not R\, x$ and offering a vote on $\{x, y\}$ neither misses an opportunity nor takes a risk at R.

Proof. Let R be a non-total proto-ranking, and suppose that for any distinct $x, y \in \mathcal{X}$ with $x \not R\, y \not R\, x$, offering a vote on $\{x, y\}$ either misses an opportunity or takes a risk at R. We shall show that R contains a missed opportunity.

Let A be the set of all pairs $\{x, y\} \subseteq \mathcal{X}$ with $x \neq y$ and $x \not R\, y \not R\, x$. The set A is non-empty since R is not total. By hypothesis, for any $\{x, y\} \in A$, offering $\{x, y\}$ either misses an opportunity or takes a risk at R, implying that some $\{z, w\} \in A$ makes $\{x, y\}$ an error at R. It follows by Lemma 8 (§E.1, p. 1) that R contains a missed opportunity. ■

E.3 Proof of Proposition 1 (p. 12)

Lemma 9. Fix a majority will W, let R be a W-reachable W-efficient ranking, and let $R' \subseteq R$ be a non-total proto-ranking containing no missed opportunities. Then there exist distinct $x, y \in \mathcal{X}$ such that $x \not R'\, y \not R'\, x$, W and R agree on $\{x, y\}$, and offering $\{x, y\}$ does not miss an opportunity or take a risk at R'.

Proof of Proposition 1. Fix a majority will W and a W-reachable W-efficient ranking R. By Proposition 2 (already proved), it suffices to find a terminal history $((x_t, y_t))_{t=1}^T$, with associated proto-rankings $(R_t)_{t=0}^T$,[40] such that

- for every $t \in \{1, \ldots, T\}$, $x_t\ W\ y_t$ and $x_t\ R\ y_t$, and
- for every $t \in \{2, \ldots, T\}$, offering $\{x_t, y_t\}$ does not miss an opportunity or take a risk at R_{t-1}.

Such a terminal history is obtained by repeatedly applying Lemma 9. ■

Proof of Lemma 9. We shall prove the contra-positive. Fix a majority will W and a W-reachable W-efficient ranking R, and let $R' \subseteq R$ be a non-total proto-ranking. Suppose that for any distinct $x, y \in \mathcal{X}$ with $x \not R'\, y \not R'\, x$ such that W and R agree on $\{x, y\}$, offering $\{x, y\}$ misses an opportunity or takes a risk at R'. We will show that R' contains a missed opportunity.

Let $\mathcal{A}$ be the set of all pairs $\{x, y\} \subseteq \mathcal{X}$ such that $x \succ y$, $x \not R'\, y \not R'\, x$, and there is no $z \in \mathcal{X}$ such that $x\ R\ z\ R\ y$. (So $\mathcal{A}$ is a set of two-element

[40] Recall that $R_0 = \varnothing$ and that R_t is the transitive closure of $\bigcup_{s=1}^t \{(x_s, y_s)\}$, for each t.

subsets of $\mathcal{X}$.) The set $\mathcal{A}$ is non-empty since it includes any R-adjacent pair $\{x, y\}$ with $x \not R' y \not R' x$, and there must be such a pair since R' is non-total and $R' \subseteq R$. By Lemma 8 (§E.1, p. 1), it suffices to show that for any pair $\{x, y\} \in \mathcal{A}$, there is a pair $\{z, w\} \in \mathcal{A}$ that makes $\{x, y\}$ an error at R'.

So fix a pair $\{x, y\} \in \mathcal{A}$. We claim that W and R must agree on $\{x, y\}$. If x, y are R-adjacent, then this holds by Observation 1 (appendix B.3) since R is W-reachable. If x, y are not R-adjacent, then since no $z \in \mathcal{X}$ satisfies $x \; R \; z \; R \; y$, it must be that $y \; R \; x$. Since $x \succ y$ and R is W-efficient, it follows that $y \; W \; x$, so that W and R agree on $\{x, y\}$.

It follows from the (contra-positive) hypothesis that offering $\{x, y\}$ either misses an opportunity or takes a risk at R'. Consider each in turn.

Case 1: $\{x, y\}$ misses an opportunity. In this case there is a $z \in \mathcal{X}$ satisfying $x \succ z \succ y$ and $y \not R' z \not R' x$. Since $\{x, y\} \in \mathcal{A}$, we must have either $z \; R \; x$ or $y \; R \; z$. Assume that $z \; R \; x$; the case $y \; R \; z$ is analogous. Since $R' \subseteq R$, we have $x \not R' z \not R' x$. Thus the pair $\{x, z\}$ lives in $\mathcal{A}$ and makes $\{x, y\}$ an error at R'.

Case 2: $\{x, y\}$ takes a risk. Assume that there is a $z \in \mathcal{X}$ such that $z \succ y$, $x \; R' \; z$ and $y \not R' z$; the case in which $x \succ z$, $z \; R' \; y$ and $z \not R' x$ is similar. Then $\{y, z\}$ makes $\{x, y\}$ an error at R'. To see that $\{y, z\}$ belongs to $\mathcal{A}$, observe that (i) $z \succ y$ and $y \not R' z$, that (ii) $z \not R' y$ since otherwise $x \; R' \; z$ and the transitivity of R' would imply the falsehood $x \; R' \; y$, and that (iii) $y \; R \; z$ since $\{x, y\} \in \mathcal{A}$ and $x \; R \; z$ (as $x \; R' \; z$ and $R' \subseteq R$), so that there is no $z' \in \mathcal{X}$ such that $z \; R \; z' \; R \; y$. ■

F Relation to ranking methods

In this appendix, we investigate the link with the social choice literature mentioned in §1.1. We recast the chair's problem as a choice among ranking methods, characterise the constraint set of this problem, and compare its solutions to ranking methods in the literature.

A *ranking method* is a map that assigns to each majority will a ranking. Each strategy σ induces a ranking method, namely the one whose value at a majority will W is the outcome of σ under W. Call a ranking method ρ *feasible* iff it is induced by some strategy, and *regret-free* iff $\rho(W)$ is W-unimprovable for every W. Clearly the chair's problem in §3 can be re-formulated as a choice between ranking methods, where the constraint set consists of the feasible ranking methods and the objective is to choose a regret-free one.

For a majority will W and rankings R, R', say that R is *more aligned with W than R'* iff for any pair $x, y \in \mathcal{X}$ of alternatives with $x \; W \; y$, if $x \; R' \; y$ then also $x \; R \; y$. This is exactly the definition in the text (§3.4), except that we allow W to be any majority will (not necessarily a ranking).

Definition 11. A ranking method ρ is *faithful* iff for every majority will W, no ranking $R \neq \rho(W)$ is more aligned with W than $\rho(W)$.

Faithfulness clearly admits a normative interpretation. It is a natural strengthening of Condorcet consistency, the requirement that $\rho(W)$ rank x highest if $x \; W \; y$ for every alternative $y \neq x$. The following shows that it also has a positive interpretation:

Observation 4. A ranking method ρ is faithful iff $\rho(W)$ is W-reachable for every majority will W.

Proof. Fix a ranking method ρ and a majority will W, and write $R \coloneqq \rho(W)$. If R is W-reachable, then any $R' \neq R$ fails to be more aligned with W since it must rank some R-adjacent pair $x \; R \; y$ as $y \; R' \; x$, where $x \; W \; y$ by Observation 1 (appendix B.3). If R is not W-reachable, then by Observation 1 there is an R-adjacent pair $x \; R \; y$ such that $y \; W \; x$, so the ranking $R' \neq R$ that agrees with R on every pair but x, y is more aligned with W. ■

By Observation 4, any feasible ranking method must be faithful. The converse does not hold, because feasibility also imposes restrictions across majority wills. To describe these constraints, we introduce a second property:

Definition 12. A ranking method ρ is *consistent* iff whenever $\rho(W) \neq \rho(W')$ for two majority wills W and W', there are alternatives $x, y \in \mathcal{X}$ such that $x \; W \; y \; W' \; x$ and

$$x \; \rho(W'') \; y \quad \text{iff} \quad x \; W'' \; y \quad \text{for every majority will } W'' \supseteq W \cap W'.$$

This property is mathematically natural, but we do not think that it has any normative appeal. Instead, it captures constraints that the rules of the interaction impose on the chair:

Proposition 4. A ranking method is feasible iff it is faithful and consistent.

Call a ranking method ρ *efficient* iff $\rho(W)$ is W-efficient for every majority will W. (W-efficiency is defined in §5.) By Theorem 2, a feasible ranking method is regret-free iff it is efficient. Thus:

Corollary 3. A ranking method is feasible and regret-free iff it is faithful, consistent and efficient.

While faithfulness has a normative interpretation, consistency and efficiency are 'positive' in nature: the former is a constraint imposed by the rules of the game, while the latter is defined in terms of the chair's self-interested preference $\succ$. This makes feasible and regret-free ranking methods quite different from those studied in the literature, which are characterised by purely normative axioms (e.g. Rubinstein (1980) for the Copeland method). Indeed, standard ranking methods such as those of Copeland and Kemeny–Slater are neither consistent nor efficient, though the latter is faithful.

Proof of Proposition 4. For necessity, let ρ be feasible. Then ρ is faithful by Observation 4. To show that it is consistent, let σ be a strategy inducing ρ, and fix majority wills W and W' such that $\rho(W) \neq \rho(W')$. Let t be the first period in which the history generated by σ and W differs from that generated by σ and W', and let $\{x, y\}$ be the pair offered in this period. Then W and W' disagree on $\{x, y\}$. Furthermore, the pair $\{x, y\}$ is clearly offered in period t of the history generated by σ and any $W'' \supseteq W \cap W'$, so that $x \; \rho(W'') \; y$ iff $x \; W'' \; y$.

For sufficiency, let ρ be faithful and consistent; we shall construct a strategy that induces ρ. For each history h, let W_h and W'_h be majority wills such that

(a) if $h = ((x_t, y_t))_{t=1}^{T}$, then $x_t \; W_h \; y_t$ and $x_t \; W'_h \; y_t$ for each $t \in \{1, \dots, T\}$, and

(b) W_h and W'_h disagree on any pair that is not voted on in h.

Since ρ is faithful, $\rho(W_h)$ is W_h-reachable and $\rho(W'_h)$ is W'_h-reachable by Observation 4. Thus by Observation 1 (appendix B.3), we have $\rho(W_h) = \rho(W'_h)$ iff h is terminal. Since ρ is consistent, we may for each non-terminal history h choose a pair $\sigma(h) \coloneqq \{x, y\} \subseteq \mathcal{X}$ that satisfies

(c) $x \; W_h \; y \; W'_h \; x$ and

(d) $x \; W'' \; y$ iff $x \; \rho(W'') \; y$ for any majority will $W'' \supseteq W_h \cap W'_h$.

Claim. Let $h = ((x_t, y_t))_{t=1}^{T}$ be a history of length $T \geq 1$ such that $\{x_1, y_1\} = \sigma(\varnothing)$ and $\{x_t, y_t\} = \sigma\Big(((x_s, y_s))_{s=1}^{t-1}\Big)$ for each $t \in \{2, \dots, T\}$. Then (i) for any majority will W'' with $x_t \; W'' \; y_t$ for each $t \in \{1, \dots, T\}$, we have $x_t \; \rho(W'') \; y_t$ for each $t \in \{1, \dots, T\}$, and (ii) the pair $\sigma(h)$ is unranked by the transitive closure of $\bigcup_{t=1}^{T} \{(x_t, y_t)\}$.

Proof of the claim. For the first part, fix a $t \in \{1, \dots, T\}$ and a majority will W'' such that $x_s \; W'' \; y_s$ for each $s \in \{1, \dots, T\}$. Define $h' \coloneqq ((x_s, y_s))_{s=1}^{t-1}$ (meaning $h' = \varnothing$ if $t = 1$), noting that $\sigma(h') = \{x_t, y_t\}$. We have $W'' \supseteq W_{h'} \cap W'_{h'}$ since $W_{h'}$ and $W'_{h'}$ satisfy (b), whence $x_t \; \rho(W'') \; y_t$ by (d).

For the second part, we have $x_t \; \rho(W_h) \; y_t$ and $x_t \; \rho(W'_h) \; y_t$ for every $t \in \{1, \dots, T\}$ by (a) and the first part of the claim, implying that $\rho(W_h)$ and $\rho(W'_h)$ (being transitive) agree on every pair ranked by the transitive closure of $\bigcup_{t=1}^{T} \{(x_t, y_t)\}$. Since $\rho(W_h)$ and $\rho(W'_h)$ disagree on the pair $\sigma(h)$ by (c) and (d), it follows that $\sigma(h)$ is unranked by the transitive closure. $\square$

By the second part of the claim, σ is a well-defined strategy.[41] To show that it induces ρ, fix a majority will W, and let $h = ((x_t, y_t))_{t=1}^{T}$ be the terminal history generated by σ and W; we must demonstrate that $\rho(W)$ is

[41] We actually defined σ only on the path. Off the path, any behaviour will do.

the transitive closure of $\bigcup_{t=1}^{T}\{(x_t, y_t)\}$. Since both are rankings, it suffices to show that $x_t\ \rho(W)\ y_t$ for every $t \in \{1, \ldots, T\}$. And this follows from the claim since $x_t\ W\ y_t$ for every $t \in \{1, \ldots, T\}$. ■

G How many *W*-reachable rankings are *W*-unimprovable?

This appendix contains two results. In §G.1, we show that for a given majority will W, every W-reachable ranking is W-unimprovable iff W is transitive. In §G.2, we show that on average across majority wills W, only a small fraction of W-reachable rankings are W-unimprovable if there are many alternatives.

G.1 When is agenda-setting valuable?

Given the majority will W, the value of agenda-setting lies in being able to reach a W-unimprovable ranking rather than some (necessarily W-reachable) ranking that is not W-unimprovable. This motivates the following definition:

Definition 13. Given her preference $\succ$ (a ranking), the chair *benefits from agenda-setting* under a majority will W iff there exists a W-reachable ranking that is not W-unimprovable.

Proposition 5. For a majority will W, the following are equivalent:

(1) W is not a ranking (i.e. is not transitive).

(2) For some $\succ$, the chair benefits from agenda-setting under W.

(3) For every $\succ$, the chair benefits from agenda-setting under W.

In words, agenda-setting is valuable precisely because it allows the chair to exploit Condorcet cycles: the chair benefits whenever there is a cycle in W, and otherwise does not benefit.

Proof. (3) immediately implies (2). To see that (2) implies (1), consider the contra-positive: if W is a ranking, then it is clearly the only W-reachable ranking, so the chair does not benefit from agenda-setting for any $\succ$.

To prove that (1) implies (3), fix any ranking $\succ$ and any majority will W that is not a ranking; it suffices to exhibit distinct W-reachable rankings R and R' such that R is more aligned with $\succ$ than R'. To that end let R be a W-efficient ranking (these are easily seen to exist). Similarly let R' be a W-*anti-efficient* ranking, i.e. one such that $x \prec y$ and $x\ W\ y$ implies $x\ R\ y$.

To show that R is more aligned with $\succ$ than R', take $x, y \in \mathcal{X}$ with $x \succ y$. If $x\ W\ y$, then $x\ R\ y$ since R is W-efficient. If instead $y\ W\ x$, then $y\ R'\ x$ since R' is W-anti-efficient. Thus $x\ R'\ y$ implies $x\ R\ y$.

It remains only to show that R and R' are distinct. Since W is not a ranking, there must be $x, y, z \in \mathcal{X}$ such that $x\ W\ y\ W\ z\ W\ x$. Suppose wlog that $x \succ z$. There are three cases. If $x \succ y \succ z$, then $x\ R\ y\ R\ z$ and $z\ R'\ x$. If $y \succ x \succ z$, then $y\ R\ z$ and $z\ R'\ x\ R'\ y$. If $x \succ z \succ y$, then $x\ R\ y$ and $y\ R'\ z\ R'\ x$. In each case, $R \neq R'$ by transitivity. ■

G.2 Most W-reachable rankings are not W-unimprovable

The following asserts that when there are enough alternatives, only a small fraction of a typical W's W-reachable rankings are W-unimprovable.

Proposition 6. For each $n \in \mathbf{N}$, let R^n (W^n) denote a uniform random draw from the set of all rankings (majority wills) on $\mathcal{X}_n \coloneqq \{1, \dots, n\}$, with R^n and W^n independent. Then

$$\Pr(R^n \text{ is } W^n\text{-unimprovable} | R^n \text{ is } W^n\text{-reachable}) \to 0 \quad \text{as } n \to \infty.$$

Proof. Fix any $n \geq 5$, and define $K_n \coloneqq \lfloor (n-1)/4 \rfloor$. Further fix a ranking R and a majority will W on $\mathcal{X}_n$, and label the alternatives $\mathcal{X}_n = \{x_1, \dots, x_n\}$ so that $x_1 \ R \ \dots \ R \ x_n$. Given $k \in \{1, \dots, K_n\}$, say that R *admits a local W-improvement at* $(x_{4k-2}, x_{4k-1}, x_{4k})$ iff both

- $x_{4k-3} \ W \ x_{4k} \ W \ x_{4k-2}$ and $x_{4k-1} \ W \ x_{4k+1}$, and
- $x_{4k} \succ x_{4k-1}$ and $x_{4k} \succ x_{4k-2}$.

If R admits a local W-improvement at $(x_{4k-2}, x_{4k-1}, x_{4k})$, then it fails to be W-unimprovable since the ranking

$$x_1 \ R' \cdots R' \ x_{4k-3} \ R' \ x_{4k} \ R' \ x_{4k-2} \ R' \ x_{4k-1} \ R' \ x_{4k+1} \ R' \cdots R' \ x_n$$

is then W-reachable (by Observation 1 in appendix B.3) and more aligned with $\succ$.

For each $n \geq 5$, let $(X_k^n)_{k=1}^n$ be random variables such that

$$\{X_1^n, \dots, X_n^n\} = \mathcal{X}_n \quad \text{and} \quad X_1^n \ R^n \ \dots \ R^n \ X_n^n \quad \text{a.s.}$$

The events '$X_{4k}^n \succ X_{4k-1}^n$ and $X_{4k}^n \succ X_{4k-2}^n$' are independent across $k \in \{1, \dots, K_n\}$ and each have probability 1/4. It follows by Observation 1 that conditional on R^n being W^n-reachable, the events

$$\text{`}R^n \text{ admits a local } W^n\text{-improvement at } (X_{4k-2}^n, X_{4k-1}^n, X_{4k}^n)\text{'}$$

are independent across $k \in \{1, \dots, K_n\}$ and have probability $(1/2)^5$. Thus

$$\Pr(R^n \text{ is } W^n\text{-unimprovable} | R^n \text{ is } W^n\text{-reachable}) \leq \left(1 - (1/2)^5\right)^{K_n},$$

which vanishes as $n \to \infty$ since $K_n = \lfloor (n-1)/4 \rfloor$ diverges. ■

H A characterisation of our 'transitive' protocol

In this appendix, we show that among all possible rules of interaction between the chair and committee that lead to a ranking, the 'transitive' protocol studied in this paper (described in §3.1) is the only one (up to restriction) that denies the chair arbitrary power and that allows votes only on pairs. This protocol is thus the natural one, given the restriction to pairwise votes. Non-binary votes raise issues that are beyond the scope of this paper.[42]

A *ballot* is a set of two or more alternatives. An *election* is (B, V), where B is a ballot and V is a map $\{1, \dots, I\} \to B$ specifying what alternative each voter votes for. An *electoral history* is a finite sequence of elections with distinct ballots. For two (distinct) electoral histories h, h', we write $h \sqsubseteq (\sqsubset)\, h'$ iff h is a truncation of h'.

A *protocol* specifies for each (permitted) electoral history either (1) a set of ballots that the chair is permitted to offer or (2) a ranking. Formally:

Definition 14. A *protocol* is $(\mathcal{H}, \rho)$, where

(1) $\mathcal{H}$ is a non-empty set of electoral histories such that

- if h' belongs to $\mathcal{H}$, then so does any $h \sqsubseteq h'$, and
- if $h = ((B_1, V_1), \dots, (B_t, V_t))$ belongs to $\mathcal{H}$, then so does $h' = ((B_1, V_1), \dots, (B_t, V_t'))$ for any $V_t' : \{1, \dots, n\} \to B_t$.

Call $h \in \mathcal{H}$ *terminal* (in $\mathcal{H}$) iff there is no $h' \sqsupset h$ in $\mathcal{H}$.

(2) ρ is a map that assigns a ranking to each terminal $h \in \mathcal{H}$.

Call an electoral history *binary* iff each ballot has exactly two elements. A *binary protocol* $(\mathcal{H}, \rho)$ is one whose $\mathcal{H}$ consists of binary electoral histories. For any binary electoral history $h = ((\{x_s, y_s\}, V_s))_{s=1}^{t}$, where wlog $|\{i : V_s(i) = x_s\}| > I/2$ for each $s \in \{1, \dots, t\}$, let R^h denote the transitive closure of $\bigcup_{s=1}^{t} \{(x_s, y_s)\}$.[43] The *transitive protocol* is the binary protocol that permits the chair to offer a ballot $\{x, y\}$ after binary electoral history h exactly if the pair x, y is unranked by R^h, and assigns the ranking R^h to each terminal h.[44]

To deny the chair arbitrary power, we focus on protocols that rank x above y whenever x won an outright majority and y was also on the ballot:

[42]Unlike in the binary case, there is no 'most natural' non-binary protocol. In particular, reasonable protocols can differ in what they deem the committee to have 'decided' in a vote on three or more alternatives in which none won an outright majority.

[43]If h is the empty electoral history, then $R^h = \varnothing$.

[44]Explicitly it is $(\mathcal{H}^\star, \rho^\star)$, where $\mathcal{H}^\star$ consists of all binary electoral histories h' such that

$$h \sqsubset ((\{x_1, y_1\}, V_1), \dots, (\{x_t, y_t\}, V_t)) \sqsubseteq h' \quad \text{implies} \quad x_t \not\!R^h\, y_t \not\!R^h\, x_t,$$

(so that $h \in \mathcal{H}$ is terminal iff R^h is a ranking,) and $\rho^\star(h) \coloneqq R^h$ for each terminal $h \in \mathcal{H}^\star$.

Definition 15. A protocol $(\mathcal{H}, \rho)$ satisfies *committee sovereignty* iff for any terminal $h = ((B_t, V_t))_{t=1}^T \in \mathcal{H}$ such that $|\{i : V_t(i) = x\}| > I/2$ and $y \in B_t \setminus \{x\}$ for some $t \in \{1, \ldots, T\}$, we have $x \; \rho(h) \; y$.

For binary protocols, committee sovereignty is equivalent to imposing transitivity after every vote:

Observation 5. A binary protocol $(\mathcal{H}, \rho)$ satisfies committee sovereignty iff $\rho(h) \supseteq R^h$ for every terminal $h \in \mathcal{H}$.

That is, any pair linked by a chain of majorities ($x \; R^h \; y$) must be ranked accordingly ($x \; \rho(h) \; y$), and so cannot be offered for a vote.[45]

Proof. Let $(\mathcal{H}, \rho)$ be binary and satisfy committee sovereignty, and take a terminal $h = ((\{x_t, y_t\}, V_t))_{t=1}^T \in \mathcal{H}$, where wlog $x_t \; R^h \; y_t$ for each $t \in \{1, \ldots, T\}$. Then $\rho(h) \supseteq \bigcup_{t=1}^T \{(x_t, y_t)\}$ by committee sovereignty, whence $\rho(h) \supseteq R^h$ because $\rho(h)$ is transitive and R^h is by definition the smallest transitive relation containing $\bigcup_{t=1}^T \{(x_t, y_t)\}$.

For the converse, let $(\mathcal{H}, \rho)$ be binary with $\rho(h) \supseteq R^h$ for every terminal $h \in \mathcal{H}$. Take any terminal $h = ((\{x_t, y_t\}, V_t))_{t=1}^T \in \mathcal{H}$ and suppose that $|\{i : V_t(i) = x_t\}| > I/2$; we must show that $x_t \; \rho(h) \; y_t$. Since $x_t \; R^h \; y_t$, this follows immediately from $\rho(h) \supseteq R^h$. ■

More is needed to deny the chair excessive power: she must also be required to offer enough ballots to give the committee a fair say. To formalise this, write $x \; S^h \; y$ for an electoral history $h = ((B_t, V_t))_{t=1}^T$ iff

$$x, y \in B_t \quad \text{and} \quad |\{i : V_t(i) = x\}| \geq |\{i : V_t(i) = y\}|$$

for some $t \in \{1, \ldots, T\}$, and say that h *gives the committee a say on* x, y iff $\{z_1, z_L\} = \{x, y\}$ for some sequence $z_1 \; S^h \; z_2 \; S^h \cdots S^h \; z_L$ of alternatives.

Definition 16. A protocol $(\mathcal{H}, \rho)$ satisfies *democratic legitimacy* iff every terminal $h \in \mathcal{H}$ gives the committee a say on each pair of alternatives.

Write $\tau(\mathcal{H})$ for the terminal elements of $\mathcal{H}$. A protocol $(\mathcal{H}, \rho)$ is a *restriction* of $(\mathcal{H}', \rho')$ iff $\tau(\mathcal{H}) \subseteq \tau(\mathcal{H}')$ and $\rho = \rho'|_{\tau(\mathcal{H})}$.[46] To wit, anything the chair can do under $(\mathcal{H}, \rho)$, she can also do under $(\mathcal{H}', \rho')$.

Proposition 7. A protocol is binary and satisfies committee sovereignty and democratic legitimacy iff it is a restriction of the transitive protocol.

[45] Formally: if $x \; R^h \; y$, then no terminal $h' \sqsupseteq h$ can feature the ballot $\{x, y\}$ (except in h). For otherwise there would be a terminal h' in which y beats x in a vote, so that $x \; R^{h'} \; y \; R^{h'} \; x$, which is impossible since $\rho(h') \supseteq R^{h'}$ and $\rho(h')$ is a ranking.

[46] $\tau(\mathcal{H}) \subseteq \tau(\mathcal{H}')$ holds exactly if $\mathcal{H} \subseteq \mathcal{H}'$ *and* any $h \in \tau(\mathcal{H})$ is terminal in $\mathcal{H}'$.

Thus any binary protocol that does not give the chair arbitrary power must be the transitive protocol, possibly with limitations on what unranked pairs the chair may offer at some histories. Neglecting such limitations as ad hoc, we arrive at the transitive protocol.

Proof. Any restriction of the transitive protocol $(\mathcal{H}^\star, \rho^\star)$ satisfies the three properties since $(\mathcal{H}^\star, \rho^\star)$ does and the properties are preserved under restriction. For the converse, let $(\mathcal{H}, \rho)$ satisfy the three properties; we must show that $\tau(\mathcal{H}) \subseteq \tau(\mathcal{H}^\star)$ and $\rho = \rho^\star|_{\tau(\mathcal{H})}$.

To establish $\tau(\mathcal{H}) \subseteq \tau(\mathcal{H}^\star)$, we show separately that $\mathcal{H} \subseteq \mathcal{H}^\star$ and that any $h \in \tau(\mathcal{H}) \subseteq \mathcal{H}^\star$ is terminal in $\mathcal{H}^\star$. For the former, fix a pair of electoral histories $h \sqsubset h' = ((\{x_s, y_s\}, V_s))_{s=1}^t \in \mathcal{H}$. We must show that the pair x_t, y_t is unranked by R^h, so suppose toward a contradiction that $x_t \; R^h \; y_t$. Then we must have $x_t \; \rho(h'') \; y_t$ for any terminal $h'' \in \mathcal{H}$ such that $h'' \sqsupseteq h$ since $\rho(h'') \supseteq R^{h''} \supseteq R^h$ by Observation 5. In particular, this must hold for any terminal $h'' \in \mathcal{H}$ with first $t-1$ elections $(\{x_1, y_1\}, V_1), \ldots, (\{x_{t-1}, y_{t-1}\}, V_{t-1})$ and t^{th} election $(\{x_t, y_t\}, V_t')$, where V_t' satisfies $|\{i : V_t'(i) = y_t\}| > I/2$. But for such an h'', committee sovereignty of $(\mathcal{H}, \rho)$ clearly demands that $y_t \; \rho(h'') \; x_t$—a contradiction.

For the latter, let $h \in \tau(\mathcal{H}) \subseteq \mathcal{H}^\star$; we must show that h is terminal in $\mathcal{H}^\star$, meaning precisely that R^h is total. Since h is binary, a pair x, y is ranked by R^h iff h gives the committee a say on x, y. And h gives the committee a say on every pair since $(\mathcal{H}, \rho)$ satisfies democratic legitimacy.

To show that $\rho = \rho^\star|_{\tau(\mathcal{H})}$, fix an $h \in \tau(\mathcal{H})$. Then $\rho(h) \supseteq R^h = \rho^\star(h)$ by Observation 5 and the definition of $\rho^\star$, and the containment must be an equality since $\rho(h)$ and $\rho^\star(h)$ are both rankings. ■

I The limits of strategic voting

In this appendix, we show that sincere voting is the unique regret-free strategy of each voter. In fact, we show something stronger: deviating from sincere voting results in a no better (a worse) outcome against any (some) strategies of the chair and the other voters, in the 'more aligned' sense.

Let each voter $i \in \{1, \ldots, I\}$ have a strict preference $\succ_i$ over the alternatives $\mathcal{X}$. A *strategy* σ_i of a voter specifies, after each history and for every offered pair x, y, whether x or y should be voted for. (Since a history records only which pairs were offered and which alternative won in each pair, not who voted how, this definition of a strategy rules out complex path-dependence. We shall relax that stricture below.)

A voter's *sincere strategy* is the one that always instructs her to vote for whichever alternative she likes better. For a strategy σ of the chair and strategies $\sigma_1, \ldots, \sigma_I$ of the voters, write $R(\sigma, \sigma_1, \ldots, \sigma_I)$ for the outcome (the ranking that results).

Definition 17. Let σ_i, σ_i' be strategies of voter i, and σ, σ_{-i} strategies of the chair and the other voters. σ_i' is *obviously better than* σ_i against σ, σ_{-i} iff $R(\sigma, \sigma_i', \sigma_{-i})$ is distinct from, and more aligned with $\succ_i$ than, $R(\sigma, \sigma_i, \sigma_{-i})$.

When one strategy is obviously better than another, it yields a better outcome no matter what voter i's exact preference over rankings, given only the weak assumption that voter i weakly prefers rankings more aligned with her preference $\succ_i$ over alternatives. By contrast, comparing strategies that are not related by 'obviously better than' involves trade-offs.

Definition 18. A strategy σ_i of a voter is *dominant* iff for any alternative strategy σ_i',

($\nexists$) there exist no strategies σ, σ_{-i} of the chair and other voters against which σ_i' is obviously better than σ_i, and

($\exists$) there exist strategies σ, σ_{-i} of the chair and other voters against which σ_i is obviously better than σ_i'.

Dominance is strong. (Albeit not as strong as conventional dominance, since 'obviously better' is only a partial ordering.) Observe that there can be at most one dominant strategy. In fact, there is exactly one:

Proposition 8. For each voter, the sincere strategy is (uniquely) dominant.

Proposition 8 remains true, with the same proof, if the definition of dominance is strengthened to allow the alternative strategy σ_i' to be an 'extended strategy' that can condition on who voted how in previous periods.

Proof. Fix a voter i, and let $\sigma_i^\star$ be her sincere strategy. We must establish properties ($\nexists$) and ($\exists$) in the definition of dominance.

Property ($\nexists$)*:* Fix strategies σ, σ_{-i} of the chair and other voters and a non-sincere strategy σ_i' of voter i, and suppose that $R' \coloneqq R(\sigma, \sigma_i', \sigma_{-i})$ is distinct from $R \coloneqq R(\sigma, \sigma_i^\star, \sigma_{-i})$; we must show that R' is not more aligned with $\succ_i$ than R. Let T be the first period in which the proto-rankings R_T and R_T' differ, and let $\{x, y\}$ be the pair voted on in this period, where (wlog) $x \ R_T \ y$ and $y \ R_T' \ x$. The two strategy profiles generate the same length-$(T-1)$ history h (by definition of T), and thus the same period-T votes $\sigma_j(h)$ by the other voters $j \neq i$. So voter i is pivotal after history h, and since $\sigma_i^\star$ is sincere it must be that $x \succ_i y$. Thus $R' \supseteq R_T'$ is not more aligned with $\succ_i$ than $R \supseteq R_T$.

Property ($\exists$)*:* Take any non-sincere strategy σ_i'. Choose strategies σ', σ_{-i}' such that σ_i' votes non-sincerely along the terminal history induced by the strategy profile $(\sigma', \sigma_i', \sigma_{-i}')$, and let T be the first period in which this occurs. Then the proto-ranking in period $T-1$ is the same under the strategy profiles $(\sigma', \sigma_i', \sigma_{-i}')$ and $(\sigma', \sigma_i^\star, \sigma_{-i}')$; call it R_{T-1}. Write $\{x, y\}$ for the pair of alternatives that are voted on in period T, where (wlog) $x \succ_i y$.

Extension claim. Let R'' be a proto-ranking, and let $x, y \in \mathcal{X}$ be distinct alternatives such that $x \not\!R'' \; y \not\!R'' \; x$. Then there is a ranking $R' \supseteq R''$ such that x, y are R'-adjacent with $x \; R' \; y$.

Proof of the extension claim. Let R'' and $x, y \in \mathcal{X}$ satisfy the hypothesis. Then $R'' \cup \{x, y\}^2$ admits a complete and transitive extension Q by the extension lemma in appendix C.2. Note that $x \; Q \; y \; Q \; x$. It follows that by appropriately breaking indifferences in Q, we may obtain a ranking $R' \supseteq R'' \cup \{(x, y)\}$ such that x, y are R'-adjacent with $x \; R' \; y$. □

By the extension claim, there exists a ranking $R \supseteq R_{T-1}$ with $x \; R \; y$ and x, y R-adjacent. Let R' be exactly R, except with the positions of x and y reversed. Clearly R is more aligned with $\succ_i$ than R', and the two are distinct.

It thus suffices to find strategies σ and σ_{-i} such that $R(\sigma, \sigma_i^\star, \sigma_{-i}) = R$ and $R(\sigma, \sigma_i', \sigma_{-i}) = R'$. For the chair, let $\sigma \coloneqq \sigma'$. As for σ_{-i}, let half of the other voters $j \in I \setminus \{i\}$ vote according R (i.e. vote for z over w iff $z \; R \; w$), and the rest vote according to R'. ■

J Extension: indecisive votes

In this appendix, we allow the vote on a pair of alternatives to be indecisive, in which case the chair may choose how they are ranked. (This occurs e.g. when the chair is a voting member of the committee.) To that end, we re-interpret $x \; W \; y$ to mean that the chair is *permitted* to rank x above y, and allow for the possibility that both $x \; W \; y$ and $y \; W \; x$. A vote on $\{x, y\}$ with $x \; W \; y$ is *indecisive* if also $y \; W \; x$, and *decisive* otherwise.

The 'majority will' W must still be total and irreflexive, but not necessarily asymmetric. By appeal to an argument similar to that for Fact 1 (appendix B.1), *any* total and irreflexive relation W should be considered.

A history still records what pairs were offered and how each pair was ranked, and a *strategy* now specifies not only what pair to offer after each history, but also how to rank them if the vote is indecisive. Note that a history does not record whether a vote was decisive or not, and thus that we rule out strategies that condition on this information. We show in supplementary appendix K how this restriction may be dropped.

Regret-free and efficient strategies are defined as before, with 'for any majority will W' replaced by 'for any total and irreflexive W'. By Lemma 1, efficiency still implies regret-freeness.

When the chair offers $\{x, y\}$ with $x \succ y$ and the vote is indecisive, we say that she *ranks in her interest* iff she ranks x above y, and *against her interest* otherwise. Augment the definition of insertion sort in §5 so that the chair ranks in her interest whenever a vote is indecisive. Theorem 1 (§5) remains true, with the same proof: insertion sort is efficient, and thus regret-free.

The characterisations of regret-free strategies (Theorems 2 and 3 in §6) extend as follows:

Theorem (2+3)′. For a strategy σ, the following are equivalent:

(a) σ is regret-free.

(b) σ is efficient.

(c) σ never misses an opportunity, takes a risk, or ranks against the chair's interest.

Proof. We establish the implications depicted in Figure 3. The proof that (c) implies (b) given in appendix C.1 applies essentially unchanged. That (b) implies (a) follows from Lemma 1.

It remains to show that (a) implies (c). To establish that regret-free strategies never miss an opportunity or take a risk, it suffices to replicate the argument in appendix C.2. To show that a regret-free strategy must not rank against the chair's interest, we prove the contra-positive. Let σ be a strategy that ranks against the chair's interest under some majority will W; we shall find a majority will W' such that the outcome R of σ under W' fails to be W'-unimprovable. In particular, we shall exhibit a W'-reachable ranking $R' \neq R$ that is more aligned with $\succ$ than R.

Let T be the first period in which σ ranks against the chair's interest under W. Write R_{T-1} for the associated end-of-period-$(T-1)$ proto-ranking, and let $\{x, y\}$ be the pair offered in period T. By hypothesis, $x \; W \; y \; W \; x$, and the chair chooses to rank y above x.

By the extension claim in supplementary appendix I (p. 13), there exists a ranking $R' \supseteq R_{T-1} \cup \{(x, y)\}$ such that x, y are R'-adjacent. Define a majority will W' by $W' := R' \cup \{(y, x)\}$, and denote by R the outcome of σ under W'. Clearly R' is W'-reachable. It remains to show that $R \neq R'$ and that R' is more aligned with $\succ$ than R.

For the former, since $x \; R' \; y$, it suffices to show that $y \; R \; x$. To this end, observe that that $R_{T-1} \subseteq R' \subseteq W'$. Thus the history of length $T-1$ generated by σ and W' is the same as that generated by σ and W, which means in particular that $\{x, y\}$ is offered in period T, and that y is ranked above x if the vote is indecisive. Under W', the vote is indeed indecisive ($x \; W' \; y \; W' \; x$), and thus $y \; R \; x$ as desired.

To show that R' is more aligned with $\succ$ than R, observe that W' agrees with R' on every pair $\{z, w\} \nsubseteq \{x, y\} = [x, y]_{R'}$. Thus by Lemma 3 in appendix C.2, R and R' agree on every pair $\{z, w\} \neq \{x, y\}$. Since $x \succ y$ and $x \; R' \; y$, it follows that R' is more aligned with $\succ$ than R. ■

All of the remaining results also extend: the characterisations of regret-freeness are tight (Propositions 1 and 2 in §6), the outcome-equivalents of insertion sort are (include) the lexicographic (amendment) strategies (Theorem 4 and Proposition 3 in §6), and sincere voting is dominant (Proposition 8 in supplementary appendix I).

K Extension: strategies with extended history-dependence

By definition, a strategy does not condition on who voted how in the past. To relax this restriction, let an *extended history* be a sequence of pairs offered and votes cast by each member on each pair, and let an *extended strategy* assign to each extended history an unranked pair of alternatives.

Recall from appendix B.1 the definition of voting profiles. The *outcome* of an extended strategy σ under a voting profile $(V_i)_{i=1}^I$ is the final ranking that results. A strategy is *regret-free* (*efficient*) iff its outcome under every voting profile $(V_i)_{i=1}^I$ is W-unimprovable (W-efficient), where W denotes the majority will of $(V_i)_{i=1}^I$.

Insertion sort is clearly an extended strategy, so is efficient by Theorem 1 (§5). Our characterisation of regret-freeness (Theorems 2 and 3 in §6) remains valid:

Theorem (2+3)″. For an extended strategy σ, the following are equivalent:

(a) σ is regret-free.

(b) σ is efficient.

(c) σ never misses an opportunity or takes a risk.

Proof. We prove the implications depicted in Figure 3. That (c) implies (b) follows from the argument in appendix C.1, which applies unchanged to extended strategies. That (b) implies (a) follows from Lemma 1.

To show that (a) implies (c), we prove the contra-positive by augmenting the argument in appendix C.2. Take an extended strategy σ that misses an opportunity or takes a risk under some voting profile $(V_i)_{i=1}^I$, and let t be the first period in which this occurs. Let W be the majority will of $(V_i)_{i=1}^I$. Construct an alternative majority will W' exactly as in the proof in appendix C.2. Construct in addition a voting profile $(V_i')_{i=1}^I$ whose majority will is W', and such that the extended history up to time t under σ and $(V_i')_{i=1}^I$ is the same as under σ and $(V_i)_{i=1}^I$. The argument in appendix C.2 ensures that the outcome of σ under $(V_i')_{i=1}^I$ fails to be W'-unimprovable. Thus σ fails to be regret-free. ■

References

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