Coalitions and Recurrent Negotiation in Multilateral Relational Contracts

Joel Watson

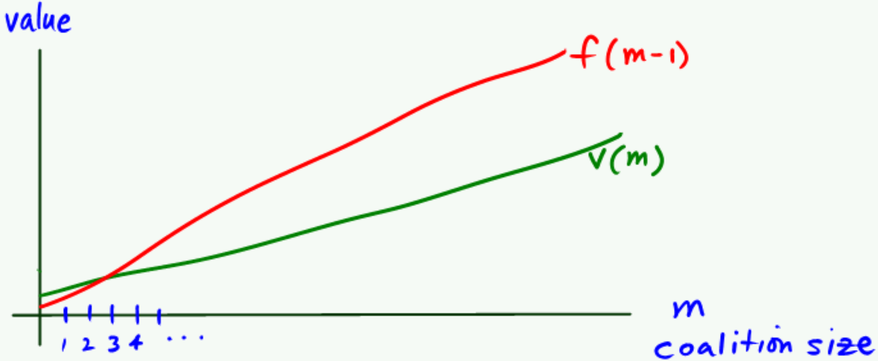
April 2024 Notes

Motivation:

- Relational contracts are ubiquitous. Many are multilateral, such as in the international arena.
- They are critical in international trade, climate policy, security arrangements, military alliances,...
- Researchers and policy makers have been interested in coalition membership, the group of parties that negotiates and manages the relational contract. (What determines its size? How effective can it be? What are the design elements?)
- A key consideration is whether and how the governing coalition can influence the behavior of parties outside the coalition, to induce them to cooperate in some way.

Standard models in the IEA literature:

- Suppose that the coalition chooses a vector of productive activity (abatement, for instance) that it commits to for every period of time.
- Outside parties best respond (free ride) and, given the coalition's commitment, myopic best response is optimal.
- Repeated game theory provides a justification for self-enforcement of the coalition's agreement. Renegotiation is sometimes included.
- In a participation game, all players individually decide whether to join the coalition. Then the coalition chooses its productive action and the outsiders myopically best respond.
- Prediction of small coalitions and minimal cooperation...



v(m) = value of member of m-size coalition, with IEA.

f(m) = value of free-riding against coalition of size m

Models described above rule out contingency agreements, such as conditioning the coalition's abatement levels on past abatement levels of outsiders in addition to those of coalition members (as needed to self-enforce the coalition's agreement anyway).

Objective: Deepen the theoretical analysis.

Approach:

- Allow full range of possible agreements on continuation play.
- Suppose the coalition's agreement serves to coordinate the play of all parties, both coalition members and outsiders.
- Take seriously that the parties themselves decide how to coordinate their behavior, through negotiation, by *modeling negotiation explicitly*.
- Model recurrent negotiation over the relational contract.
- Build on MW 2013, which incorporates an explicit account of contract negotiation in repeated game with a *transfer phase*, with a theory of agreement and disagreement that selects among the PPE.

Output:

- Logic regarding myopic behavior by outside parties.
- Implications of bargaining power.
- Conditions for more successful coalitions.

1 Basic Model Without Coalitions

Repeated game with transfer phase

Player set $\{1, 2, \ldots, n\} \equiv N$, discrete period, infinite horizon, common discount factor δ.

Within each period:

- Bargaining and transfer phase
  - Simultaneous voluntary transfers; player i: $m \in \mathbb{R}^n_-(i) \equiv \{m \in \mathbb{R}^n_- \mid m_i \leq 0\}$.
  - Transfers are observed. Total transfer vector $m \in \mathbb{R}^n_-$.
- Action/production phase
  - Stage game: $\langle A, u \rangle$, $A = A_1 \times A_2 \times \cdots A_n$, $u\colon A \to \mathbb{R}^n$.
  - Action profile is observed.
  - Normalize the payoffs by $1 - \delta$.

Payoff in the period: $(1 - \delta)u(a) + m$. Allow for public randomization device.

International climate agreement application:

- Each player is a country, $a_i \in A_i = [0, 1]$ is country i's abatement effort, and u_i is country i's welfare.
- Assume u_i is increasing in a_j for all $j \neq i$. Suppose $u_i(0, a_{-i}) > u_i(a_i, a_{-i})$ for all $a_i > 0$, so that $a_i = 0$ corresponds to country i abating at the level best for its own welfare only. Zero abatement is each country's myopic optimum.

Recursive characterization of equilibrium continuation values

(minor extension of APS 1990, focusing on pure-strategy equilibrium here)

- Consider $y\colon A \to \mathbb{R}^n$ that gives the continuation value from the next period as a function of the realized action profile $a \in A$ in the current period (embodies how players coordinate their behavior in the future).
- If $\tilde{a} \in A$ is a Nash equilibrium of $\langle A, (1-\delta)u + \delta y \rangle$ then y is said to *enforce* $\tilde{a}$, and the continuation value from the current period is $w = (1-\delta)u(\tilde{a}) + \delta y(\tilde{a})$.
- If $W \subset \mathbb{R}^n$ is the set of achievable continuation values from the start of the next period, then the set of continuation values supported from the production phase (stage game) in the current period is:

$$D(W) \equiv \left\{(1-\delta)u(\tilde{a}) + \delta y(\tilde{a}) \,\middle|\, \exists\, y\colon A \to \text{co } W \text{ and } \tilde{a} \in A \;\text{ s.t. } y \text{ enforces } \tilde{a}\right\}.$$

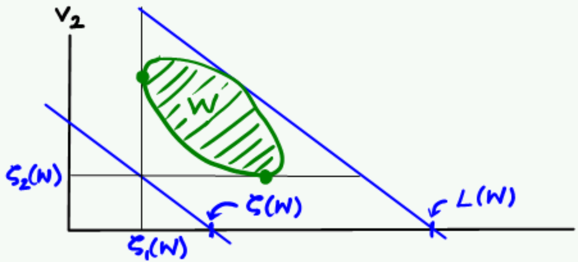
- If $W \subset \mathbb{R}^n$ is a compact set of achievable continuation values from the production phase (stage game) in the current period, then the set of continuation values supported from the bargaining/transfer phase in the current period is:

$$\text{tri } W \equiv \left\{ w \in \mathbb{R}^n \mid w_j \geq \zeta_j(W) \text{ for all } j \in N, \; \sum_{j \in N} w_j \leq L(W) \right\},$$

where

$$\zeta_i(W) \equiv \min_{w \in W} w_i \text{ and } L(W) \equiv \max_{w \in W} \sum_{i \in N} w_i \text{ (called the level of } W\text{).}$$

- In the repeated game, the set of subgame-perfect equilibrium values is the largest set $X \subset \mathbb{R}^n$ such that $X \subset \text{tri } D(X)$.



- Let $\hat{u}_i(a) \equiv \max_{a'_i \in A_i} u_i(a'_i, a_{-i})$.
- Because we have assumed actions are observed and we are limiting attention to pure-strategy equilibrium, we have the following:

  Assuming W is closed, $w \in D(W)$ if and only if there exist $\tilde{a} \in A$ and $v \in \text{co } W$ such that $w = (1-\delta)u(\tilde{a}) + \delta v$ and $(1-\delta)[\hat{u}_i(\tilde{a}) - u_i(\tilde{a})] \leq \delta(v_i - \zeta_i(W))$ for every $i \in N$.

  ($v \in \Delta W$ because the players can use the public randomization device to mix over W in the next period.)

Theory of relational contracting (contractual equilibrium, MW 2013)

(explicit account of negotiation on continuation play, to select continuation value)

Negotiation theory:

- Bargaining protocol in the bargaining phase (followed by voluntary transfers or externally enforced transfers); unanimity requirement for agreement.
- Agreement axiom: If an offer to switch to a credible continuation value is accepted, continuation play achieves this value.
- Disagreement theory (no fault): Disagreement continuation value does not depend on the manner of disagreement, but can depend on history; promised transfers seal the deal (failure to transfer triggers disagreement).

Noncooperative and hybrid versions…

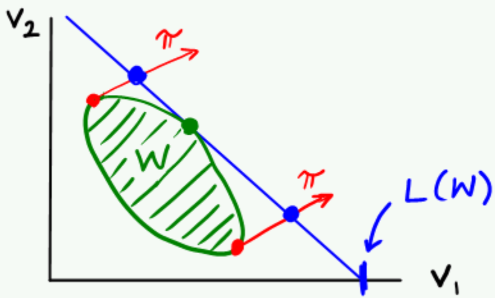
Transfers and bargaining provide useful structure.…

Recursive characterization of contractual equilibrium values

- Exogenous bargaining weights $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ represent the bargaining protocol.
- If $W \subset \mathbb{R}^n$ is the set of equilibrium continuation values from the production phase (stage game) of the current period, then the set of equilibrium continuation values from the bargaining phase is:

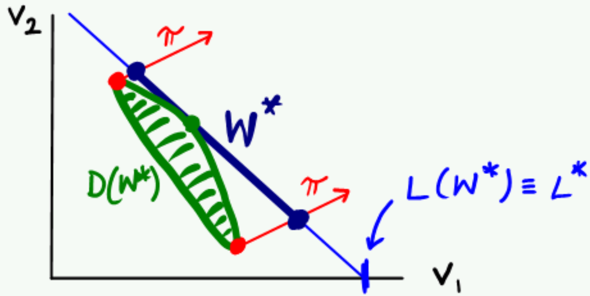
$$B(W) \equiv \left\{ \underline{w} + \pi \left(L(W) - \sum_{i \in N} \underline{w}_i \right) \,\middle|\, \underline{w} \in W \right\}.$$

(Disagreement point $\underline{w}$ entails no immediate transfer, and can depend on the history of play in prior periods.)



V2
π
π
W
L(W)
V1

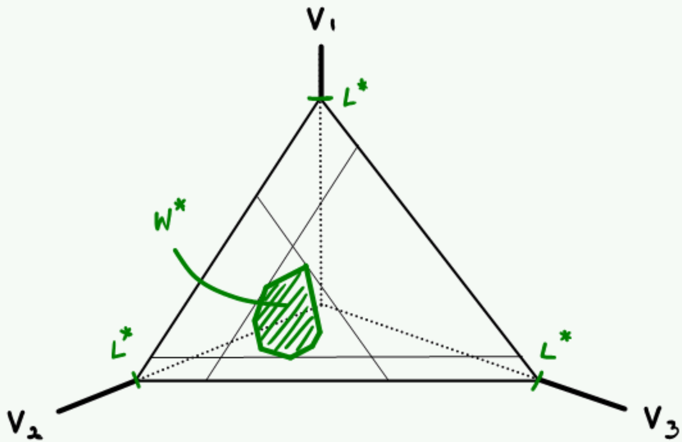
- The set of contractual equilibrium values (CEV), denoted by W^*, is the maximal fixed point of $B(D(\cdot))$, having the highest level L^*.

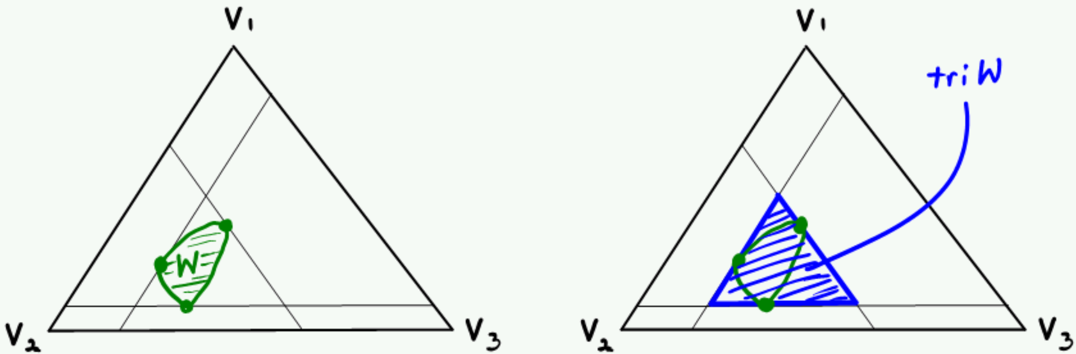


General notes:

- Existence... Monotone property of operator $B(D(\cdot))$ in a level-normalized sense (think of points in $D(\cdot)$ and negotiate to a level; or combine fixed points).
- W^* and L^* depend on bargaining weights.
- General theme: Bargaining power arises in any theory of coordination based on negotiation that refines the equilibrium set (MW 2023).

- For two-player games, a simple algorithm calculates the CEV set.
- It is more difficult to compute for n-player settings generally.
- However, a straightforward algorithm exists for two-round stage games with production followed by voluntary transfers, assuming observable productive actions and restriction to pure-strategy equilibrium.




V1
V2
V3
W
V1
V2
V3
tri W

Aside: Algorithm to calculate CEV set

Assumption 1: The stage game has simultaneous observed productive-action choices ($e_i \in [0,1]$ for player i) followed by simultaneous voluntary transfers, so that $a_i = (e_i, \tau_i)$ where $\tau_i \colon [0,1]^n \to \mathbb{R}^n_-(i)$ and $u_i(a) = \mu_i(e) + \sum_{j \in N} \tau_j(e)_i$. Limit attention to pure-strategy equilibria. Each function μ_i is continuous, and $\langle [0,1]^n, \mu \rangle$ has a Nash equilibrium in pure strategies.

Define: $\zeta(W) \equiv \sum_{i \in N} \zeta_i(W)$, $\hat{U}(a) \equiv \sum_{i \in N} \hat{u}_i(a)$, and $U(a) \equiv \sum_{i \in N} u_i(a)$.

For player i to not gain by deviating from action profile a, $(1-\delta)\left[\hat{u}_i(a) - u_i(a)\right] \leq \delta d_i$ is required, where d_i is the punishment in terms of reduced continuation value from the next period.

Sum constraint: $(1-\delta)\left[\hat{U}(a) - U(a)\right] \leq \delta d$, validity to be determined…

Theorem: Under Assumption 1, contractual equilibrium (in pure strategies) exists. The CEV set W^* has the following properties:

- First,

$$L(W^*) = \max_a U(a)$$

$$\text{subject to } \; a \in A \; \text{ and } \; (1-\delta)\left[\hat{U}(a) - U(a)\right] \leq \delta d^*,$$

$$\text{where } \; d^* \equiv L(W^*) - \zeta(W^*).$$

- Second, d^* is the largest fixed point of $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$, where $\gamma(d) \equiv \sum_{i \in N} \gamma_i(d)$ and

$$\gamma_i(d) \equiv \max_a \pi_i U(a) - \hat{u}_i(a)$$

$$\text{subject to } \; a \in A \; \text{ and } \; (1-\delta)\left[\hat{U}(a) - U(a)\right] \leq \delta d.$$

Optimal allocation of bargaining power

Theorem: Under Assumption 1 and considering $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ as a design parameter, the highest contractual equilibrium joint value L is attained by giving all bargaining power to one player (setting $\pi_i = 1$ for some player i).

Conclusions for an IEA application:

- The optimal allocation of bargaining power is to the country i^* with the largest fixed point of $\gamma_i(\cdot; 1)$.

$$\gamma_i(d; 1) \equiv \max_{a \in A} \pi_i U(a) - \hat{u}_i(a) \text{ subject to } (1-\delta)\left[\hat{U}(a) - U(a)\right] \leq \delta d.$$

- Country i^* serves as a "residual claimant" with regard to bargaining surplus.
- In a period in which country i^* is to be punished (the others rewarded), in disagreement the others choose relatively high abatement levels, which maximizes their joint value except for the distortion caused by $\hat{u}_{i^*}(a)$ and subject to the collective incentive condition.
- In a period in which a country $j \neq i^*$ is to be punished (country i^* rewarded), in disagreement all countries other than j choose minimal abatement and country j chooses the maximal abatement this country can be given the incentive to select.

Example:

- $u_i(a) = b_i \sum_{j \neq i} a_j - c_i a_i^2$, where parameters $b_i > 0$ and $c_i >$ measure marginal benefit of other countries' abatement and marginal cost of own abatement.
- If countries have similar costs but country i^* has significantly lower marginal benefit than the others, then it is best to give country i^* the most bargaining power possible. (Intuition: γ_{i^*} objective function is high and has less distortion than for other countries, good for holding i^*'s continuation value down.)
- If countries have similar benefits but country i^* has significantly lower marginal cost than the others, then it is best to give country i^* the most bargaining power possible. (Intuition: With small marginal cost, country i^* can be given the incentive to abate at high levels, which is most effective for punishing this player.)
- Bargaining power may be associated with these institutional features: the right to make offers, the right to set the agenda, etc.

2 Model With Coalitions

Extending the theory to analyze contracting by a stable coalition:

- A subset of players negotiates an agreement.
- What does the agreement specify? At least the behavior of coalition members, which can be conditioned on actions taken by players outside the coalition.
- Outsiders are also rational and therefore best respond. So the coalition's agreement at least indirectly serves to coordinate the play of parties outside the coalition.
- In this theory, coordinated behavior is the result of active contracting and no other institution. Outsiders lack any other way of coordinating.
- Thus, the coalition negotiates to select the behavior of all players.
- And there is renegotiation.

Coalition relational contracting (Coalition CE)

Suppose a coalition of players $C \subset N$ (the participants) forms at the beginning, it is fixed over time, and these players negotiate to determine how all of the players coordinate their behavior. The repeated-game layout is the same as before.

Negotiation theory:

- Fixed bargaining weights $(\pi_i)_{i \in C} \equiv \pi_C$ represent the bargaining protocol.
  $\pi_i \geq 0$, $\sum_{i \in C} \pi_i = 1$.
- In the bargaining phase, transfers between coalition members only (or assume players ignore external transfers).
- Select SPE that satisfy *agreement* and *disagreement conditions*, as before.
- Players outside the coalition play their part of the plan that the coalition selects, in accordance with sequential rationality.

Recursive characterization of coalition CEV

- If W is the set of continuation values from the beginning of the next period, then $D(W)$ is the set of equilibrium values from the production phase of the current period.
- For any compact set $W \subset \mathbb{R}^n$, define $L^C(W) \equiv \max_{w \in W} \sum_{i \in C} w_i$ and

$$\Lambda^C(W) \equiv \left\{ w \in W \;\middle|\; \sum_{i \in C} w_i = L^C(W) \right\}.$$

- Bargaining outcome: Suppose $W \subset \mathbb{R}^n$ is the set of achievable continuation values from the production phase in the current period, and in disagreement the players would coordinate on play to obtain $\underline{w} \in W$.

  The coalition will negotiate to instead get some value $w \in \Lambda^C(W)$ and make an immediate transfer so that the value for coalition members is

$$\underline{w}_C + \pi_C \left(L^C(W) - \textstyle\sum_{i \in N} \underline{w}_i \right).$$

- If $W \subset \mathbb{R}^n$ is the set of achievable continuation values from the production phase (stage game) in the current period, then the set of equilibrium continuation values from the bargaining phase is:

$$B^C(W) \equiv \left\{ \left(\underline{w}_C + \pi_C \left(L^C(W) - \textstyle\sum_{i \in C} \underline{w}_i \right), \, w_{-C} \right) \,\middle|\, \underline{w} \in W, \, (w_C, w_{-C}) \in \Lambda^C(W) \right\}.$$

- The set of coalition CEV, denoted by W^*, is the maximal fixed point of $B^C(D(\cdot))$, having the highest level L^{C*}.
- We don't yet have a broad existence result in this setting (fixed points exist; the issue is highest level and monotonicity), but we have existence assuming some structure on the stage game, such as the assumption made below.

Motivating players outside the coalition to cooperate

If contracting occurred only in period 1 and there were no opportunity to renegotiate, then outsiders would be induced to cooperate under the optimal relational contract.

Opportunities to renegotiate can undermine cooperation by the contracting parties.

New element here: Opportunities to renegotiate also undermine the ability of contracting parties to induce those outside the coalition to cooperate. *In fact, it can be worse than the constraint on cooperation within the coalition.*

To get a benchmark result, put structure on the stage game. A simple version:

Assumption 2: Action space A_i is convex, for every $i \in N$. Further, u_i is bounded and strictly concave, and $\hat{u}_i - u_i$ is convex, for every $i \in N$.

Theorem: Under Assumption 2, given any coalition C, the coalition CEV set W^* exists and has the property that $\{w_{-C} \mid w \in W^*\}$ is a singleton. In equilibrium, players outside the coalition myopically best respond in each period.

Proof:

- In any period, negotiation leads to some value in $W^* = B^C(D(W^*))$ consisting of an immediate transfer within the coalition plus some continuation value $w \in D(W^*)$ from the current-period production phase.
- Note that w is achieved by coordinating on an action profile $\tilde{a} \in A$ in the current period, leading to a continuation value $v \in \mathrm{co}\ W^*$ from the next period, such that

$$w = (1-\delta)u(\tilde{a}) + \delta v \in \Lambda^C(D(W^*)) \tag{1}$$

and for every $i \in N$,

$$(1-\delta)[\hat{u}_i(\tilde{a}) - u_i(\tilde{a})] \leq \delta(v_i - \zeta_i(W)). \tag{2}$$

- Claim: There is a unique $\tilde{a}$ that can be enforced using continuation values in W^* and that yields $w \in \Lambda^C(D(W^*))$. Presume not. Then we can find $\tilde{a}, \tilde{a}' \in A$ and $v, v' \in \text{co } W^*$, such that conditions 1 and 2 hold for both $(\tilde{a}, v)$ and $(\tilde{a}', v')$, yielding continuation values w and w' respectively. It is the case that $(v + v')/2 \in \text{co } W^*$. By convexity of $\hat{u}_i - u_i$, condition 2 holds for $((\tilde{a} + \tilde{a}')/2, (v + v')/2)$ and so $(\tilde{a} + \tilde{a}')/2$ is enforced by $(v + v')/2$. This implies that $(1 - \delta)u((\tilde{a} + \tilde{a}')/2) + \delta(v + v')/2 \in D(W^*)$, but by strict concavity of u_i for every $i \in N$, the joint value to coalition C must strictly exceed the joint values of w and w' (which are the same for C), contradicting that w and w' are in $\Lambda^C(D(W^*))$.

- Because players outside the coalition do not take part in negotiation and transfers in the bargaining phase, their equilibrium value from the start of any period is the weighted average of $u(\tilde{a})$ in the current period and a continuation value in the convex hull of W^* from the next period. That is, for every $w \in W^*$, there is a value $v \in \operatorname{co} W^*$ such that $w_i = (1-\delta)u_i(\tilde{a}) + \delta v$ for every $i \notin C$. Because $\delta < 1$ and W^* must be bounded, this implies that $\{w_i \mid w \in W^*\}$ is a singleton.
- As a consequence, there is no way to enforce anything other than myopically optimal actions in the stage game for players outside the coalition.

Realistic departures from the grim conclusion

A few comments. . .

Sequential-move stage game: Two-round setting discussed earlier (a round of productive actions followed by a round of transfers)

- Even with outsiders behaving myopically within a period, they can be given the incentive to play non-myopically in the production round by the promise of contingent payments from coalition members in the second round.
- Essentially short-term contracts between coalition members and outsiders…
- The coalition enforces these short-term contracts, such as punishing a coalition member who fails to make a promised external transfer.
- Outsiders break even in these arrangements due to the coalition's coordination power.
- Analysis provides a foundation for, and expands on, discussion of "trade restrictions" in the literature on international agreements…

Extension: random negotiation opportunity

- In the bargaining phase, with probability ρ the coalition has an opportunity to negotiate. Otherwise members can still make voluntary transfers.
- Coalition CEV set W^* is a fixed point of $\rho B^C(D(\cdot)) + (1-\rho)\,\mathrm{tri}^C\, D(\cdot)$.
- It is the case that $\rho B^C(D(\cdot)) + (1-\rho)\,\mathrm{tri}^C\, D(\cdot)$ is monotone decreasing in ρ, so the level is weakly higher for lower ρ.
- Essentially can make longer-term implicit contracts with external players…

These factors increase the difference between the value of being in the coalition and the value of being outside, increasing pressure to join the coalition.

But coalition membership is of secondary importance. Cooperative behavior, internal and external, is the primary interest.

3 Thoughts about Modeling Endogenous Coalitions

A model we have thought about, but have not fully analyzed:

- At the beginning of a period, there is an existing coalition C from the previous period.
- The set of equilibrium continuation values depends on C, $W(C)$.
- Given the history, and if no agreement is reached, the players would coordinate on behavior to achieve disagreement value $\underline{w} \in D(W(C))$.

- Coalition C negotiates to make an ultimatum offer to the players needed to form any coalition C', to play so as to achieve continuation value $\hat{w} \in D(W(C'))$, and make immediate transfers.
- Each player $i \in C$ will get $w_i = \underline{w}_i + \pi_i^C \sum_{j \in C \cup C'} (\hat{w}_j - \underline{w}_j)$.
- Each player $i \in C' \backslash C$ gets $\underline{w}_i$, and each player $i \notin C \cup C'$ gets $\hat{w}_i$.
- Coalition C chooses C' and $\hat{w}$ to maximize $\sum_{j \in C \cup C'} (\hat{w}_j - \underline{w}_j)$.
- This defines B as a function of C and $\mathcal{W} = (W(C))_{C \in 2^N \backslash \{\emptyset\}}$.

Looking for a stability result on the coalition…

Thanks.

Let me know if this is a useful modeling exercise and where it can go from here.

More details: Algorithm to calculate CEV set

Assumption: The stage game has simultaneous observed productive-action choices ($e_i \in [0,1]$ for player i) followed by simultaneous voluntary transfers, so that $a_i = (e_i, \tau_i)$ where $\tau_i\colon [0,1]^n \to \mathbb{R}^n_-(i)$ and $u_i(a) = \mu_i(e) + \sum_{j \in N} \tau_j(e)_i$. Limit attention to pure-strategy equilibria. Each function μ_i is continuous, and $\langle [0,1]^n, \mu \rangle$ has a Nash equilibrium in pure strategies.

Define: $\zeta(W) \equiv \sum_{i \in N} \zeta_i(W)$, $\hat{U}(a) \equiv \sum_{i \in N} \hat{u}_i(a)$, and $U(a) \equiv \sum_{i \in N} u_i(a)$.

For player i to not gain by deviating from action profile a, $(1-\delta)\left[\hat{u}_i(a) - u_i(a)\right] \leq \delta d_i$ is required, where d_i is the punishment in terms of reduced continuation value from the next period.

Sum constraint: $(1-\delta)\left[\hat{U}(a) - U(a)\right] \leq \delta d,$ validity to be determined…

Theorem: Under the assumption above, contractual equilibrium (in pure strategies) exists. The CEV set W^* has the following properties:

- First,

$$
\begin{aligned}
L(W^*) = \max_a \; & U(a) \\
& \text{subject to } \; a \in A \;\text{ and }\; (1-\delta)\left[\hat{U}(a) - U(a)\right] \leq \delta d^*, \\
& \text{where } \; d^* \equiv L(W^*) - \zeta(W^*).
\end{aligned}
$$

- Second, d^* is the largest fixed point of $\gamma \colon \mathbb{R}_+ \to \mathbb{R}_+$, where $\gamma(d) \equiv \sum_{i \in N} \gamma_i(d)$ and

$$
\begin{aligned}
\gamma_i(d) \equiv \max_a \; & \pi_i U(a) - \hat{u}_i(a) \\
& \text{subject to } \; a \in A \;\text{ and }\; (1-\delta)\left[\hat{U}(a) - U(a)\right] \leq \delta d.
\end{aligned}
$$

Sketch of proof of the first theorem:

- As discussed earlier, for a compact set W, $w \in D(W)$ if and only if there exist $\tilde{a} \in A$ and $v \in W$ such that $w = (1-\delta)u(\tilde{a}) + \delta v$ and $(1-\delta)[\hat{u}_i(\tilde{a}) - u_i(\tilde{a})] \leq \delta(v_i - \zeta_i(W))$ for every $i \in N$. (That a is not fully observed does not matter to the argument, because any unilateral deviation that changes the outcome of the stage game would be detected and punished severely.)
- Without loss of generality, we can assume that $\tilde{a} = (\tilde{e}, \tilde{\tau})$ satisfies $\tilde{\tau}_i(e) = 0$ for every $e \neq \tilde{e}$. That is, if any player deviates by not choosing their part of $\tilde{e}$, then $\tilde{a}$ specifies that no one make a transfer in the second round of the stage game (the best punishment within the stage game).

- We have $w = (1-\delta)u(\tilde{a}) + \delta v = (1-\delta)[\mu(\tilde{e}) + \sum_{j \in N} \tilde{\tau}_j(\tilde{e})] + \delta v$ and $\hat{u}_i(\tilde{a}) = \max\{\hat{\mu}_i(\tilde{e}),\ \mu_i(\tilde{e}) + \sum_{j \neq i} \tilde{\tau}_j(\tilde{e})_i\}$, where $\hat{\mu}_i(\tilde{e}) = \max_{e_i} \mu_i(e_i, \tilde{e}_{-i})$. The first element of the maximum expression is the value of deviating by choosing the myopically optimal productive action, which would be followed by no transfers, whereas the second element is the value of deviating only by not making the specified transfer in the second round.

- It must be that $D(W) = D(\text{tri } W)$. Here is the argument: Note that $\zeta_i(\text{tri } W) = \zeta_i(W)$ for every i. Consider any $w \in D(\text{tri } W)$ and let $\tilde{a} \in A$ and $v \in \text{tri } W$ be such that the stage-game incentive condition holds for every player. Take any $v' \in W$. We can find transfer vectors $m'_1 \in \mathbb{R}^n_0(1), \ldots, m'_n \in \mathbb{R}^n_0(n)$ with the following properties: first, $m'_i = 0$ if $v'_i \leq v_i$, and otherwise $m'_{ii} = -(v'_i - v_i)$, for every $i \in N$; second, $\sum_{i \in N} m'_i = v' - v$. Define τ' so that $\tau'_i(\tilde{e}) = \tilde{\tau}_i(\tilde{e}) + \frac{\delta}{1-\delta} m'_i$ and $\tau'_i(e) = 0$ for every i and $e \neq \tilde{e}$. Let $a' = (\tilde{e}, \tau')$. By construction, $(1-\delta)u(a') + dev' = w$ and one can verify that the incentive condition $(1-\delta)[\hat{u}_i(a') - u_i(a')] \leq \delta(v'_i - \zeta_i(W))$ holds for every player i.

- Using the fact that $W^* = B(D(W^*)) = B(D(\text{tri } W^*))$, we have

$$\zeta_i(W^*) = \min_{a,v} (1-\delta)u_i(a) + \delta v_i + \pi_i \left[L(W^*) - (1-\delta)U(a) - \delta \sum_{j \in N} v_j \right]$$

$$\text{subject to } \; a \in A, \; v \in \text{tri } W^*, \; \text{and}$$

$$(1-\delta)\left[\hat{u}_j(a) - u_j(a)\right] \leq \delta(v_j - \zeta_j(W^*)) \text{ for } j \in N.$$

- The solution to the minimization problem entails player i's incentive constraint binding and $\sum_{j \in N} v_j = L(W^*)$, for otherwise player i extracts extra bargaining surplus which would raise the value of the objective function.
- Also, the sum incentive constraint is necessary and sufficient for the individual ones, given $v \in \text{tri } W^*$.
- The minimization problem therefore simplifies to

$$\zeta_i(W^*) = \min_a (1-\delta)\hat{u}_i(a) + \delta\zeta_i(W^*) + \pi_i(1-\delta)L(W^*) - \pi_i(1-\delta)U(a)$$

$$\text{subject to } \; a \in A \; \text{ and } \; (1-\delta)\left[\hat{U}(a) - U(a)\right] \leq d^*.$$

- Subtracting $\delta\zeta_i(W^*) + \pi_i(1-\delta)L(W^*)$ and dividing by $(1-\delta)$, we obtain

$$\zeta_i(W^*) - \pi_i L(W^*) = \min_a \hat{u}_i(a) - \pi_i U(a)$$
$$\text{subject to } \; a \in A \; \text{ and } \; (1-\delta)\left[\hat{U}(a) - U(a)\right] \leq d^*.$$

- Reversing the sign yields $\pi_i L(W^*) - \zeta_i(W^*) = \gamma_i(d^*)$ and summing over i establishes the fixed-point property.
- The characterization of $L(W^*)$ results from the fact that the maximization problem shown in the Theorem identifies the highest joint value that can be achieved in a period, and this identifies d^* as the largest fixed point of γ.

More details: Optimal allocation of bargaining power

Theorem: Under the assumption above and considering $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ as a design parameter, the highest contractual equilibrium joint value L is attained by giving all bargaining power to one player (setting $\pi_i = 1$ for some player i).

Logic for the second theorem:

- Recall that L^* is positively related to d^* and that d^* is the largest fixed point of $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\gamma(d) = \sum_{i=1}^{n} \gamma^i(d)$, where

$$\gamma_i(d) \equiv \max_a \ \pi_i U(a) - \hat{u}_i(a)$$
$$\text{subject to} \ \ a \in A \ \ \text{and} \ \ (1-\delta)\left[\hat{U}(a) - U(a)\right] \leq \delta d.$$

- Write $\gamma_i(d;\pi_i)$ and $\gamma(d;\pi)$ to show the dependence on parameter π.
- Standard arguments establish that $\overline{d} \equiv \max_\pi\{d^* \mid d^* = \gamma(d^*;\pi)\}$ exists.
- Lemma: $\gamma_i(d;\pi_i)$ is strictly increasing and convex in π_i.

- Suppose there is no π satisfying $\overline{d} = \gamma(\overline{d}; \pi)$ and $\pi_i = 1$ for some i. Let $\overline{\pi}$ be any value that satisfies $\overline{d} = \gamma(\overline{d}; \overline{\pi})$. Then from the Lemma there exist players i and j such that $\overline{\pi}_i, \overline{\pi}_j \in (0, 1)$ and

$$\gamma_i(\overline{d}; \overline{\pi}_i + \overline{\pi}_j) + \gamma_j(\overline{d}; 0) \geq \gamma_i(\overline{d}; \overline{\pi}_i) + \gamma_j(\overline{d}; \overline{\pi}_j).$$

  Redefine $\overline{\pi}$ so that player i's bargaining weight becomes $\overline{\pi}_i + \overline{\pi}_j$ and player j's bargaining weight is 0.

- Repeat the previous step until we arrive at a vector $\hat{\pi}$ that satisfies $\hat{\pi}_i = 1$ for some i and $\hat{\pi}_j = 0$ for all $j \neq i$. By construction, $\gamma(\overline{d}; \hat{\pi}) \geq \overline{d}$, implying that $\gamma(\cdot; \hat{\pi})$ has a fixed point weakly larger than $\overline{d}$, which contradicts what we assumed above.