Lecture 2: Multiple Long-Run Players and Reputational Bargaining

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Last Lecture: Reputation with *One* Long-Run Player

Model: One long-run player *vs* a sequence of short-run players.

Reputation: The long-run player could be one of the several commitment types $\alpha_1^* \in \Omega^m$ who plays some exogenous strategy.

Results: When the long-run player is sufficiently patient, his equilibrium payoff is at least

$$\max_{\alpha_1^* \in \Omega^m} \min_{\alpha_2 \in B_0(\alpha_1^*)} u_1(\alpha_1^*, \alpha_2).$$

and his equilibrium payoff is at most

$$\sup_{\alpha_1 \in \Delta(A_1)} \max_{\alpha_2 \in B_0(\alpha_1)} u_1(\alpha_1, \alpha_2).$$

Intuition: In every period, either the short-run players play a best reply to type ω's equilibrium action, or the probability they assign to type ω goes up after observing type ω's equilibrium action.

Last Lecture: Reputation with *One* Long-Run Player

The patient player's equilibrium payoff is at least

$$\max_{\alpha_1^* \in \Omega^m} \min_{\alpha_2 \in B_0(\alpha_1^*)} u_1(\alpha_1^*, \alpha_2).$$

The patient player's equilibrium payoff is at most

$$\sup_{\alpha_1 \in \Delta(A_1)} \max_{\alpha_2 \in B_0(\alpha_1)} u_1(\alpha_1, \alpha_2)$$

If his actions are statistically identified, then $B_0(\alpha_1^*) = \mathrm{BR}_2(\alpha_1^*)$.

- If (u_1, u_2) is generic and Ω^m contains the patient player's optimal commitment action, then

$$\max_{\alpha_1^* \in \Omega^m} \min_{a_2 \in B_0(\alpha_1^*)} u_1(\alpha_1^*, a_2) = \sup_{\alpha_1 \in \Delta(A_1)} \max_{\alpha_2 \in B_0(\alpha_1)} u_1(\alpha_1, \alpha_2).$$

- **Takeaway:** Reputation effects lead to sharp predictions on payoffs.

This Lecture: Multiple Long-lived Players

Myopic players:

- Once they are convinced that their opponents will play some action, they will play a best reply against that action.

Forward-looking players:

- Their opponent can convince them that he will play some commitment action with high prob.
- Their opponent can also convince them that he will play some commitment action with high prob in the next 100 periods.
- But will the forward-looking player play a stage-game best reply?

Can a patient player secure a high payoff against a patient opponent?

Cripps and Thomas (1997)

Example: Suppose players' actions can be perfectly monitored.

-	L	R
T	$1, 1$	$0, 0$
B	$0, 0$	$0, 0$

Both players' discount factors are δ.

With prob π_0, P1 is committed and plays T at every history.

With prob $1 - \pi_0$, P1 is the rational type.

Theorem: Cripps and Thomas (1997)

For every $\varepsilon > 0$*, there exist* $\overline{\pi} > 0$ *and* $\underline{\delta} \in (0, 1)$ *s.t. for all* $\pi_0 < \overline{\pi}$ *and* $\delta > \underline{\delta}$*, there exists a sequential equilibrium in which P1's payoff* $< \varepsilon$*.*

Why? How general is this lesson?

Chan (2000): Folk Theorem in Reputation Games

Let $\underline{v}_i$ be player i's minmax value, and let $\overline{v}_i$ be player i's highest feasible payoff conditional on player j's payoff is at least $\underline{v}_j$.

Failure of reputation effects besides two classes of games.

1. Dominant Action Games:

   If there exists $a_1^* \in A_1$ such that

   (a) a_1^* is a strictly dominant action for P1,
   (b) $u_1(a_1^*, a_2) = \overline{v}_1$ for every $a_2 \in \text{BR}_2(a_1^*)$.

2. Strictly Conflicting Interests:

   There exists $a_1^* \in A_1$ such that for every $a_2 \in \text{BR}_2(a_1^*)$,

   $$u_1(a_1^*, a_2) = \overline{v}_1 \text{ and } u_2(a_1^*, a_2) = \underline{v}_2.$$

   For every $a \equiv (a_1, a_2) \in A_1 \times A_2$, if $u_1(a) = \overline{v}_1$, then $u_2(a) = \underline{v}_2$.

Folk Theorem in Chan (2000)

Folk Theorem in Reputation Games (Chan 2000)

If the stage game belongs to none of these categories, then for every feasible and strictly individually rational payoff of P1, there exist $\overline{\pi} > 0$ *and* $\underline{\delta} \in (0,1)$ *such that when the probability of commitment type is less than* $\overline{\pi}$ *and the discount factor is greater than* $\underline{\delta}$*, there exists a sequential equilibrium in which the rational-type player* 1 *obtains this payoff.*

Constructive Proof: Overview of Equilibrium Strategies

Length of the learning phase $N \in \mathbb{N}$ and mixing prob $\{\phi_t\}_{t=0}^{N-1}$.

For every $t \in \{0, ..., N-1\}$, the rational type P1 plays $\phi_t T + (1-\phi_t)B$ if T has been played in all previous periods.

If P1 has played T from 0 to $N-1$, then

- play (T, L) forever starting from period N.

In period 0 to $N-1$, if P2 has not observed B, then she plays R.

If P1 plays B for the first time in period $t \leq N-1$ and $a_{2,t} = R$,

- Continuation play in period $t+1$ delivers payoff δ^{N-1-t}.

If P1 plays B for the first time in period $t \leq N-1$ and $a_{2,t} = L$,

- Continuation play in period $t+1$ delivers payoff 0.

Constructive Proof: Idea

From P1's perspective:

- He needs to suffer for N periods in order to obtain the reward 1.
- He can end the suffering at any time by revealing rationality.
- The earlier he ends the suffering, the smaller reward he receives.
- In equilibrium, he is indifferent between sustaining his reputation and ending the suffering at any time from 0 to $N-1$.

From P2's perspective:

- She knew that L is optimal in the stage game.
- But why does she play R from period 0 to $N-1$?
- The fear of being punished in the future if she plays L while P1 plays B.

Tradeoff between Learning and Incentive Provision

Question: Do there exist mixing prob $\{\phi_t\}_{t=0}^{N-1}$ and N that work?

- P1's prob of playing B must be large enough to deter P2 to play L.
- P1's prob of playing B must be small enough to slow down learning.
- We need N to be large enough s.t. players receive low payoff.
- We need N to be small enough s.t. P1's reputation in period N does not exceed 1.

Key step of proof: Construct $\{\phi_t\}_{t=0}^{N-1}$ and N s.t.

1. ϕ_t is small enough s.t. P2 has an incentive to play R.
2. ϕ_t is large enough and N is small enough s.t. P2's belief about the commitment type is below 1 after observing T from period 0 to $N-1$.
3. N is large enough so that $1-\delta^N$ is close to 1.

Incentive Constraints & Learning

For every $t \in \{0, 1, ..., N-1\}$

- π_t: Prob of commitment type after observing T from 0 to $t-1$.
- P2's payoff if he plays R: δ^{N-t}

  P2's payoff if he plays L: $(\pi_t + (1-\pi_t)\phi_t)(1-\delta+\delta^{N-t})$
- P2's incentive constraints implies:

$$\pi_t + (1-\pi_t)\phi_t \leq \frac{\delta^{N-t}}{1-\delta+\delta^{N-t}}.$$

- Bayes Rule suggests that:

$$\pi_{t+1} = \frac{\pi_t}{\pi_t + (1-\pi_t)\phi_t} \quad \Leftrightarrow \quad \underbrace{\pi_t + (1-\pi_t)\phi_t}_{\text{prob of } T \text{ in period } t} = \frac{\pi_t}{\pi_{t+1}}$$

- Suppose ϕ_t is just small enough s.t. IC binds, then $\pi_N < 1$ iff:

$$\Pi_{\tau=0}^{N-1} \frac{\delta^{N-\tau}}{1-\delta+\delta^{N-\tau}} = \frac{\pi_0}{\pi_N} > \pi_0.$$

Existence of π_0 and N

Remaining task: Can we find $\pi_0 \in (0,1)$ such that for every δ close to 1, there exists N such that:

$$\Pi_{\tau=0}^{N-1} \frac{\delta^{N-\tau}}{1-\delta+\delta^{N-\tau}} > \pi_0 \tag{1}$$

and

$$\delta^N < \varepsilon. \tag{2}$$

This is not trivial since

- The first inequality requires N to be small enough.
- The second inequality requires N to be large enough.

Existence of π_0 and N

Let's work with the LHS of the first inequality:

$$\Pi_{\tau=0}^{N-1}\frac{\delta^{N-\tau}}{1-\delta+\delta^{N-\tau}}.$$

Taking logs and use $\log x \geq 1 - 1/x$ for all $x \in (0,1)$, we have:

$$\sum_{\tau=0}^{N-1}\log\frac{\delta^{N-\tau}}{1-\delta+\delta^{N-\tau}} > \sum_{\tau=0}^{N-1}\left\{1-\frac{1-\delta+\delta^{N-\tau}}{\delta^{N-\tau}}\right\} = -(1-\delta)\sum_{\tau=0}^{N-1}\delta^{\tau-N}.$$

$$= \delta - \delta^{-N}.$$

Hence,

$$\Pi_{\tau=0}^{N-1}\frac{\delta^{N-\tau}}{1-\delta+\delta^{N-\tau}} > \pi_0$$

is implied by $\delta - \delta^{-N} > \log \pi_0$.

Existence of π_0 and N

Hence, it is sufficient to find $\pi_0 \in (0,1)$ such that for every δ close to 1, there exists N such that:

$$\delta - \delta^{-N} > \log \pi_0,$$

and

$$\delta^N < \varepsilon.$$

Choose $\pi_0 \in (0,1)$ to be small enough such that

$$\log \pi_0 < 2(\delta - \frac{1}{\varepsilon}).$$

- If N is such that $\delta^N \approx \varepsilon$, then $\delta - \delta^{-N} > \log \pi_0$.

Reputation Games with Multiple Long-Run Players

Cripps and Thomas (1997) and Chan (2000):

- In general, reputation effects cannot lead to sharp predictions when all players are long-lived.

Rubinstein (1982) and the ensuing bargaining literature:

- It is hard to incorporate incomplete information.
- The predictions are sensitive to the bargaining protocol.

Next: Dividing a dollar bargaining game.

- Reputation effects lead to sharp predictions.
- The predictions are robust to a large class of bargaining protocols.

Review: Rubinstein Bargaining Game

Two players decide how to divide a dollar.

- Time: $t = 0, \Delta, 2\Delta, ...$ Player i's discount factor $\delta_i \equiv e^{-r_i\Delta}$.

  Interpret Δ as period length and r_i as player i's interest rate.

In period $2k\Delta$, P1 makes an offer $\alpha_1 \in [0,1]$.

- If P2 accepts, then the game ends.

  Payoffs: $\alpha_1\delta_1^{2k}$ for player 1, and $(1-\alpha_1)\delta_2^{2k}$ for player 2.
- If P2 rejects, then the game moves on to the next period.

In period $(2k+1)\Delta$, P2 makes an offer $\alpha_2 \in [0,1]$.

- If P1 accepts, then the game ends.

  Payoffs: $(1-\alpha_2)\delta_1^{2k+1}$ for player 1, and $\alpha_2\delta_2^{2k+1}$ for player 2.
- If P1 rejects, then the game moves on to the next period.

Rubinstein's Theorem

Theorem: Bargaining under Complete Information (Rubinstein 82)

There exists a unique subgame perfect equilibrium.

On the equilibrium path, an agreement is reached in period 0.

Player 1*'s payoff is* $\frac{1-\delta_2}{1-\delta_1\delta_2}$. *Player* 2*'s payoff is* $\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$.

As the bargaining friction vanishes, i.e., $\Delta \to 0$

Player 1's payoff converges to:

$$\lim_{\Delta\to 0} \frac{1-e^{-r_2\Delta}}{1-e^{-(r_1+r_2)\Delta}} = \frac{r_2}{r_1+r_2}.$$

Player 2's payoff converges to:

$$\lim_{\Delta\to 0} \frac{e^{-r_2\Delta}(1-e^{-r_1\Delta})}{1-e^{-(r_1+r_2)\Delta}} = \frac{r_1}{r_1+r_2}.$$

We call $\left(\frac{r_2}{r_1+r_2}, \frac{r_1}{r_1+r_2}\right)$ players' Rubinstein bargaining payoffs.

Issues with Rubinstein's Result

How to incorporate incomplete information?

- Existing literature: One-sided incomplete info, only one player can make offers, etc.

The predictions are sensitive to the bargaining protocol.

Introducing Incomplete Info to Rubinstein Bargaining

Player i is rational with prob $1 - z_i$.

Player i is committed with prob z_i.

- a set of bargaining postures $C_i \equiv \{\alpha_i^1, \alpha_i^2, ..., \alpha_i^{k_i}\} \subset [0, 1]$
- with prob $z_i \pi_i(\alpha_i^j)$, always demands α_i^j, and accepts iff receives $\geq \alpha_i^j$.

$\pi_i(\alpha_i^1) + \pi_i(\alpha_i^2) + ... + \pi_i(\alpha_i^{k_i}) = 1.$

Question: How will players behave and what is the division of surplus?

Lesson from 80s: Directly solving this game is hard.

Abreu and Gul (2000)'s Approach

Three steps:

1. Continuous-time war-of-attrition with one commitment type for each player. Each player either mimics the commitment type or concedes.
2. Extend the results by allowing for multiple commitment types. Which commitment type will the rational type imitate?
3. In reputational bargaining games, when players can make offers frequently ($\Delta \to 0$), revealing rationality $\approx$ conceding to opponent.

   When offers are frequent, players' payoffs in the reputational bargaining game $\approx$ their payoffs in a war-of-attribution game.

Payoffs in the reputational bargaining game $\approx$ Rubinstein payoffs when

- offers are frequent,
- commitment types occur with low probability and players' commitment probabilities are comparable,
- the set of commitment types is rich enough.

War-of-Attrition with One Commitment Type on Each Side

Two players decide how to divide a dollar.

- Time $t \in [0, +\infty)$. Players' interest rates r_1 and r_2.
- With prob z_i, player i is committed, demands α_i^*, and never concedes.
- With prob $1 - z_i$, player i is rational and chooses $\widetilde{t}_i \in [0, +\infty]$,
  where $\widetilde{t}_i$ is the time at which player i concedes,
- The game ends at $\widetilde{t} \equiv \min\{\widetilde{t}_1, \widetilde{t}_2\}$.
- We assume that $\alpha_1^* + \alpha_2^* > 1$.
- The rational types' payoffs:
  * if $\widetilde{t}_1 > \widetilde{t}_2$, then $\alpha_1^* e^{-r_1 \widetilde{t}}$ for P1 and $(1 - \alpha_1^*) e^{-r_2 \widetilde{t}}$ for P2.
  * if $\widetilde{t}_1 < \widetilde{t}_2$, then $(1 - \alpha_2^*) e^{-r_1 \widetilde{t}}$ for P1 and $\alpha_2^* e^{-r_2 \widetilde{t}}$ for P2.
  * if $\widetilde{t}_1 = \widetilde{t}_2$, then share the surplus equally.

Mixed Strategy in Continuous Time War-of-Attrition

Rational-type of player i's mixed action can be represented by:

- a distribution of their concession time $\widetilde{F}_i(\cdot) \in \Delta[0, +\infty]$.

We will work with $F_i(\cdot) \equiv (1 - z_i)\widetilde{F}_i(\cdot)$.

- $F_i(\cdot)$ is the unconditional distribution of player i's concession time.

  $F_i(t)$ is the prob that player i concedes before or at time t.

  This is what their opponent cares about.

- $F_i(t) \in [0, 1 - z_i]$ for every finite t.

  If $F_i(t) = 1 - z_i$, then player i has a perfect reputation at time t.

Equilibrium Construction

We construct an equilibrium with the following features:

1. At most one player concedes with positive prob at time 0.
2. The rational types of both players finish conceding at the same time τ.
3. Both players concede at a constant rate before time τ.

Later on, we will show that this is the unique equilibrium.

Let us pin down the values of:

1. Players' concession rates when $t \in (0, \tau]$.
2. The time at which concession stops τ.
3. Who concedes with positive prob at time 0 (if any), with what prob?

Equilibrium Construction: Compute Concession Rates

Player i's concession rate at t:

$$\lambda_i(t) \equiv \left| \frac{d(1 - F_i(t))/dt}{1 - F_i(t)} \right|.$$

Player j is indifferent between conceding at $t \in (0, \tau)$ and conceding at the next time instant:

$$\lambda_i(t) \underbrace{\Big(\alpha_j^* - (1 - \alpha_i^*)\Big)}_{\text{player } j\text{'s gain if player } i \text{ concedes}} = \underbrace{r_j(1 - \alpha_i^*)}_{\text{player } j\text{'s cost of waiting}}.$$

This yields the expression for the equilibrium concession rate:

$$\lambda_i(t) = \frac{(1 - \alpha_i^*) r_j}{\alpha_i^* + \alpha_j^* - 1}.$$

Since the above expression is independent of t, we write λ_i instead of $\lambda_i(t)$.

For every $t \in [0, \tau]$,

$$1 - F_i(t) = \Big(1 - F_i(0)\Big) e^{-\lambda_i t}.$$

Compute τ & Who Concedes in At Time 0

Suppose nobody concedes with positive prob at time 0,

- Let T_i be the time it takes for player i to build a perfect reputation:

$$e^{-\lambda_i T_i} = z_i,$$

or equivalently,

$$T_i = -\frac{\log z_i}{\lambda_i}.$$

If $T_1 = T_2$, then nobody concedes with positive prob at 0.

- $\tau = T_1 = T_2$

If $T_i > T_j$, then $\tau = T_j$ and player i concedes with positive prob at time 0 s.t.

$$\Big(1 - \underbrace{F_i(0)}_{\text{concession prob at 0}}\Big)e^{-\lambda_i T_j} = z_i \quad \Rightarrow \quad F_i(0) = 1 - z_i z_j^{-\lambda_i/\lambda_j}$$

Both players finish conceding at the same time if player i concedes with probability $F_i(0)$ at time 0.

Lessons from this equilibrium

Equilibrium payoffs when player i concedes with positive prob at $t = 0$:

- Player i's payoff is $1 - \alpha_j^*$.
- Player j's payoff is $(1 - \alpha_i^*)(1 - F_i(0)) + \alpha_j^* F_i(0)$.

The strength of player i increases in his rate of reputation building

$$\lambda_i \equiv \frac{r_j(1 - \alpha_i^*)}{\alpha_i^* + \alpha_j^* - 1},$$

and increases in his initial commitment probability z_i.

A player is *stronger* if:

- he is more patient than his opponent,
- his commitment demand is less greedy,
- and he is more likely to be the commitment type.

The Uniqueness of Equilibrium

We establish some necessary conditions for equilibrium:

1. At most one player concedes with positive prob at time 0.

   Otherwise, one player strictly prefers to wait for another instant.

2. The rational type of every player concedes in finite time.

   If i doesn't concede at t, then i expects j to concede before $t + T$ with positive prob. If j does not concede, j's prob of committed goes up.

3. Both players stop conceding at the same time.

   No incentive to wait when the other player will never concede.

4. Both players concede at a constant rate when $t \in (0, \tau]$.

   Key step: F_1 and F_2 must be continuous and strictly increasing.

   The indifference conditions for every $t \in (0, \tau]$ yield the unique rate.

Smooth & Positive Concession from 0 to τ

Lemma

$F_1(t)$ and $F_2(t)$ are continuous and strictly increasing when $t \in (0, \tau)$.

1. If F_1 jumps at t, then F_2 does not jump at t.

   This is because P2 can benefit from waiting at t.

2. If F_1 is constant on $[t', t'']$, then F_2 is also constant on $[t', t'']$.

   For P2, conceding at (t', t'') strictly dominated by conceding at t'.

3. $\nexists$ interval $[t', t''] \subset [0, \tau]$ s.t. both F_1 and F_2 are constants.

   Let t^* be the largest t'' s.t. F_1 and F_2 are constants on $[t', t'']$.

   Since F_1 and F_2 cannot both jump at t^*, either P1 or P2's payoff is continuous at t^*. Let's say P1's payoff is continuous at t^*.

   P1's payoff from conceding at $t' + \varepsilon >$ conceding at $t^* - \varepsilon$, by continuity at t^*, also $>$ conceding at $t^* + \varepsilon$, contradicting def of t^*.

Smooth & Positive Concession from 0 to τ

Lemma

$F_1(t)$ and $F_2(t)$ are continuous and strictly increasing when $t \in (0, \tau)$.

1. For every $t \in (0, \tau]$, if F_1 jumps at t, then F_2 does not jump at t.
2. If F_1 is constant on $[t', t'']$, then F_2 is also constant on $[t', t'']$.
3. $\nexists$ interval $[t', t''] \subset [0, \tau]$ s.t. both F_1 and F_2 are constants.
4. 2 and 3 implies that F_1 and F_2 are strictly increasing on $[0, \tau]$.
5. Why are both F_1 and F_2 continuous?

   If F_1 jumps at t, then F_2 is constant on $(t - \varepsilon, t)$, contradicting 4.

Implication of this lemma:

- Both players are indifferent from 0 to τ.
- Their indifference conditions pin down their concession rates.

Multiple Commitment Types: Who to Imitate?

Let $C_i \subset [0,1]$ be a finite set of commitment types.

- z_i: prob of player i is committed.
- $\pi_i(\alpha_i^*)$: Prob of committing to $\alpha_i^* \in C_i$ *conditional on* i is committed.

$t = -1$: rational type announces which commitment type to imitate.

Simplifying assumption: Transparent commitment types.

- can be relaxed when commitment types are stationary.
- important when commitment types are nonstationary (Wolitzky 11).

Players' Payoffs

There exists a unique equilibrium. Why?

- P1's incentive to take a bargaining posture becomes weaker when P2's belief about P1 taking that bargaining posture increases.

Interesting limit: Fix all other parameters and take $(z_1, z_2) \to 0$.

- A sequence of commitment probabilities: $\{z_1^n, z_2^n\}_{n=1}^{\infty}$.
- v_i^n: Player i's equilibrium payoff in game (z_1^n, z_2^n).

Theorem: War-of-Attrition with Rich Set of Commitment Types

If $\lim z_1^n = \lim z_2^n = 0$ *and* $\liminf \frac{z_1^n}{z_1^n + z_2^n}, \limsup \frac{z_1^n}{z_1^n + z_2^n} \in (0,1)$, *then:*

$$\liminf_{n\to\infty} v_i^n \geq \max\left\{\alpha_i^* \in C_i \ s.t.\ \alpha_i^* \leq \frac{r_j}{r_i + r_j}\right\}.$$

Implication: If C_i is sufficiently rich, then player i can approximately secure their Rubinstein bargaining payoff $\frac{r_j}{r_i+r_j}$.

A Heuristic Explanation

Fix (α_1^*, α_2^*), player i's concession time is:

$$T_i \approx -\frac{\log z_i^n}{\lambda_i} = -\frac{(\alpha_i^* + \alpha_j^* - 1)\log z_i^n}{r_j(1-\alpha_i^*)}.$$

Player 1 is stronger when $T_1 < T_2$ and vice versa.

Ratio between T_1 and T_2:

$$\frac{T_1}{T_2} \approx \frac{r_1(1-\alpha_2^*)}{r_2(1-\alpha_1^*)} \times \underbrace{\frac{\log z_1^n}{\log z_2^n}}_{\text{goes to 1 as } z_1^n \text{ and } z_2^n \text{ go to 0 at the same rate}}.$$

Consider player 1's payoff by imitating commitment type $\alpha_1^* \leq \frac{r_2}{r_1+r_2}$.

- If $\alpha_2^* \leq 1-\alpha_1^*$, then player 1 receives α_1^* in period 0.
- If $\alpha_2^* > 1-\alpha_1^*$, then $T_1 < T_2$ when n is large enough $\Rightarrow$ P1 is strong.

A Heuristic Explanation

When $\alpha_1^* \leq \frac{r_2}{r_1+r_2}$ and $\alpha_2^* > 1 - \alpha_1^*$, $T_1 < T_2$ when n is large enough.

- Why? $\frac{r_2(1-\alpha_1^*)}{\alpha_1^*+\alpha_2^*-1} = \lambda_1 > \lambda_2 = \frac{r_1(1-\alpha_2^*)}{\alpha_1^*+\alpha_2^*-1}$.
- When $z \to 0$, it takes longer to build reputation, so T_1/T_2 depends only on the ratio between players' concession rates.

The weak player (player 2)'s concession prob at time 0:

$$F_2(0) = 1 - z_2 z_1^{-\lambda_2/\lambda_1}.$$

Compute the term $z_2 z_1^{-\lambda_2/\lambda_1}$ as $n \to \infty$.

- When $\lim z_1^n/z_2^n$ is bounded, $\lim z_1^n = 0$, and $\lim z_2^n = 0$,

  $z_2 z_1^{-\lambda_2/\lambda_1}$ goes to 0 for every fixed (λ_1, λ_2) with $\lambda_1 > \lambda_2$.

Therefore, $F_2(0) \approx 1$ as $n \to +\infty$.

Players' Guaranteed Payoffs and Equilibrium Payoffs

Recap: By committing to the Rubinstein bargaining payoff $\frac{r_2}{r_1+r_2}$,

- P1 guarantees payoff $\frac{r_2}{r_1+r_2}$ when $\alpha_2^* \leq \frac{r_1}{r_1+r_2}$.
- As $n \to \infty$, P1's payoff is approximately $\frac{r_2}{r_1+r_2}$ when $\alpha_2^* > \frac{r_1}{r_1+r_2}$ since P2's concession prob at time 0 is close to 1.

Similarly, P2 can guarantee payoff $\approx \frac{r_1}{r_1+r_2}$ by demanding $\frac{r_1}{r_1+r_2}$.

Since players' Rubinstein payoffs lie on the Pareto frontier, this approximately pins down both players' equilibrium payoffs.

From War-of-Attrition to Bargaining

Each player picks a bargaining posture, and decides when to concede.

- Next: What if each player can flexibly choose what to offer in an alternating offer bargaining game?

Important insight: Reveal rationality $\approx$ conceding when offers are frequent.

Lemma

$\forall\, \varepsilon > 0$, $\exists\, \overline{\Delta} > 0$, *s.t. when* $\Delta < \overline{\Delta}$, *at every history* h^t *s.t.*

- *P1 has revealed rationality*
- *P2 hasn't separated from commitment type* α_2^*,

then P1's payoff $\leq 1 - \alpha_2^* + \varepsilon$, *and P2's payoff* $\geq \alpha_2^* - \varepsilon$.

A Heuristic Explanation

After revealing rationality, P1 will concede in finite time with prob 1.

Let T be the last time P1 concedes. We show that $T \to 0$ as $\Delta \to 0$.

- Suppose P1 has the option to concede at $T - \Delta$ but he does not.
- His incentive not to concede implies that P2 will accept his offer at $T - \Delta$ with positive prob, denoted by π.
- At time $T - \Delta$, P2 gets $\alpha_2^* e^{-r\Delta}$ by waiting, so she will not accept any offer that gives her less than $\alpha_2^* e^{-r\Delta}$.
- P1's incentive constraint at $T - \Delta$:

$$\pi \underbrace{(1 - \alpha_2^* e^{-r\Delta})}_{\text{the most P1 can get if P2 accepts his offer}} + (1 - \pi)(1 - \alpha_2^*)e^{-r\Delta} \geq 1 - \alpha_2^*,$$

A Heuristic Explanation

- Let π be the prob that P2 accepts P1's offer at $T - \Delta$.
- P1's incentive constraint at $T - \Delta$:

$$\pi \underbrace{(1 - \alpha_2^* e^{-r\Delta})}_{\text{the most P1 can get if P2 accepts his offer}} + (1 - \pi)(1 - \alpha_2^*)e^{-r\Delta} \geq 1 - \alpha_2^*.$$

This inequality implies that

$$\pi \geq 1 - \alpha_2^*.$$

- Hence, P2's reputation is multiplied by $\frac{1}{1-\alpha_2^*}$ within Δ units of time.
- Do the same exercise for time $T - 2\Delta$, $T - 3\Delta$, $T - 4\Delta$,...
- As $\Delta \to 0$, P2's reputation goes to 1 almost instantaneously.

Robustness to Bargaining Protocols

Consider a general reputational bargaining game.

- $t \in [0, +\infty)$.
- Bargaining protocol $g : [0, +\infty) \to \{0, 1, 2, 3\}$,

  $g(t) = 0$: no one can make offer at t.

  $g(t) = 1$: only P1 can make offer at t.

  $g(t) = 2$: only P2 can make offer at t.

  $g(t) = 3$: both players offer simultaneously at t.
- Assumptions:
  1. each player makes infinitely many offers from 0 to $+\infty$.
  2. each player makes finitely many offers at any bounded interval.
- Summarize the bargaining game by its bargaining protocol g.

Convergence Result

Definition: Convergence to Continuous Time

$\{g_n\}_{n=1}^{\infty}$ converges to continuous time if for every $\varepsilon > 0$, there exists $\overline{n}$ s.t. for all $n \geq \overline{n}$, $t \geq 0$, and $i \in \{1, 2\}$, there exists $\widehat{t} \in [t, t+\varepsilon]$ such that $i = g_n(\widehat{t})$.

Only requires each player can make at least one offer in any ε-interval.

- Allows for many ways to approach continuous time.

Benchmark without Commitment Types

Suppose $\{g_n\}_{n=1}^{\infty}$ converges to continuous time. Let σ_n be a sequential equilibrium in g_n, and $(v_{1,n}, v_{2,n})$ be players' payoffs in σ_n, then $\lim_{n\to\infty} v_{i,n}$ is player i's payoff in continuous-time war-of-attrition.

Continuous-time war-of-attrition captures what happens when players can make offers frequently.

- Not sensitive to the ways of approaching continuous time.